

THE COMPLEX VOLUME OF $\mathrm{SL}(n, \mathbb{C})$ -REPRESENTATIONS OF 3-MANIFOLDS

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ABSTRACT. For a compact 3-manifold M with arbitrary (possibly empty) boundary, we give a parametrization of the set of conjugacy classes of boundary-unipotent representations of $\pi_1(M)$ into $\mathrm{SL}(n, \mathbb{C})$. Our parametrization uses Ptolemy coordinates, which are inspired by coordinates on higher Teichmüller spaces due to Fock and Goncharov. We show that a boundary-unipotent representation determines an element in Neumann's extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$, and use this to obtain an efficient formula for the Cheeger-Chern-Simons invariant, and in particular for the volume. Computations for the census manifolds show that boundary-unipotent representations are abundant, and numerical comparisons with census volumes, suggest that the volume of a representation is an integral linear combination of volumes of hyperbolic 3-manifolds. This is in agreement with a conjecture of Walter Neumann, stating that the Bloch group is generated by hyperbolic manifolds.

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1. INTRODUCTION

For a closed 3-manifold M , the Cheeger-Chern-Simons invariant [6, 7] of a representation ρ of $\pi_1(M)$ in $\mathrm{SL}(n, \mathbb{C})$ is given by the Chern-Simons integral

$$(1.1) \quad \widehat{c}(\rho) = \frac{1}{2} \int_M s^* (\mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)) \in \mathbb{C}/4\pi^2\mathbb{Z},$$

where A is the flat connection in the flat $\mathrm{SL}(n, \mathbb{C})$ -bundle E_ρ with holonomy ρ , and $s: M \rightarrow E_\rho$ is a section of E_ρ . Since $\mathrm{SL}(n, \mathbb{C})$ is 2-connected a section always exists, and a different choice of section changes the value of the integral by a multiple of $4\pi^2$.

When $n = 2$, the imaginary part of the Cheeger-Chern-Simons invariant equals the hyperbolic volume of ρ . More precisely, if $D: \widetilde{M} \rightarrow \mathbb{H}^3$ is a developing map for ρ and $\nu_{\mathbb{H}^3}$ is the hyperbolic volume form, $\mathrm{Im}(\widehat{c}(\rho))$ equals the integral of $D^*(\nu_\rho)$ over a fundamental domain for M . In particular, if $M = \mathbb{H}^3/\Gamma$ is a hyperbolic manifold, and ρ is a lift to $\mathrm{SL}(2, \mathbb{C})$ of the geometric representation $\rho_{\mathrm{geo}}: \pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$, the imaginary part equals the volume of M . In fact, in this case we have

$$(1.2) \quad \widehat{c}(\rho) = i(\mathrm{Vol}(M) + i\mathrm{CS}(M)),$$

where $\mathrm{CS}(M)$ is the Chern-Simons invariant of M (with the Riemannian connection). Although this result is known to experts, no proof seems to be available (see [8, 18] for discussions). We give a proof in Section 2. The invariant $\mathrm{Vol}(M) + i\mathrm{CS}(M)$ is often referred to as *complex volume*. Motivated by this, we define the complex volume $\mathrm{Vol}_{\mathbb{C}}$ of a representation $\rho: \pi_1(M) \rightarrow \mathrm{SL}(n, \mathbb{C})$ by

$$(1.3) \quad \widehat{c}(\rho) = i \mathrm{Vol}_{\mathbb{C}}(\rho)$$

and define the *volume* of ρ to be the real part of the complex volume, i.e. the imaginary part of the Cheeger-Chern-Simons invariant. Surprisingly, as we shall see, the relationship to hyperbolic volume seems to persist even when $n > 2$.

The set of $\mathrm{SL}(n, \mathbb{C})$ -representations is a complex variety with finitely many components, and the complex volume is constant on components. This follows from the fact that representations in the same component have cohomologous Chern-Simons forms. Hence, for any M , the set of complex volumes is a finite set.

We show that the definition of the Cheeger-Chern-Simons invariant naturally extends to compact manifolds with boundary, and representations $\rho: \pi_1(M) \rightarrow \mathrm{SL}(n, \mathbb{C})$ that are *boundary-unipotent*, i.e. take peripheral subgroups to a conjugate of the unipotent group N of upper triangular matrices with 1's on the diagonal. We formulate all our results in this more general setup.

The main result of the paper is a concrete algorithm for computing the set of complex volumes. The idea is that the set of (conjugacy classes of) boundary-unipotent representations can be parametrized by a variety, called the *Ptolemy variety*, which is defined by homogeneous polynomials of degree 2. The Ptolemy variety depends on a choice of triangulation, but if the triangulation is sufficiently fine, every representation is detected by the Ptolemy variety. A point c in the Ptolemy variety naturally determines an element $\lambda(c)$ in Neumann's extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$, and if ρ is the representation corresponding to c , we have

$$(1.4) \quad R(\lambda(c)) = i \mathrm{Vol}_{\mathbb{C}}(\rho),$$

where $R: \widehat{\mathcal{B}}(\mathbb{C}) \rightarrow \mathbb{C}/4\pi^2\mathbb{Z}$ is a Rogers dilogarithm.

There is a canonical group homomorphism

$$\phi_n: \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SL}(n, \mathbb{C})$$

coming from the natural $\mathrm{SL}(2, \mathbb{C})$ -action on the vector space $\mathrm{Sym}^{n-1}(\mathbb{C}^2)$. The map ϕ_n preserves unipotent elements, and we show that composing a boundary-unipotent representation in $\mathrm{SL}(2, \mathbb{C})$ with ϕ_n multiplies the complex volume by $\binom{n+1}{3}$. If $M = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold, the geometric representation ρ_{geo} always lifts to a representation in $\mathrm{SL}(2, \mathbb{C})$, but if M has cusps, no such lift is boundary-unipotent. In fact, by a result of Calegari [5], any lift takes a longitude to an element with trace -2 . When n is even, we shall thus, more generally, be interested in boundary-unipotent representations in

$$(1.5) \quad p\mathrm{SL}(n, \mathbb{C}) = \mathrm{SL}(n, \mathbb{C})/\langle \pm I \rangle.$$

Such representations have a complex volume defined modulo $\pi^2 i$, and our algorithm computes these as well. By studying representations in $p\mathrm{SL}(n, \mathbb{C})$, we make sure that when M is hyperbolic, there is always at least one representation with non-trivial complex volume, namely $\phi_n \circ \rho_{\mathrm{geo}}$.

Walter Neumann has conjectured that every element in the Bloch group $\mathcal{B}(\mathbb{C})$ is an integral linear combination of Bloch group elements of hyperbolic 3-manifolds. Since the extended Bloch group equals the Bloch group up to torsion, Neumann's conjecture would imply that all complex volumes are, up to rational multiples of $i\pi^2$, integral linear combinations of complex volumes of hyperbolic 3-manifolds. In particular, the volumes should all be integral linear combinations of volumes of hyperbolic manifolds.

Our algorithm has been implemented by Matthias Goerner. The algorithm uses Magma [3] to compute a Groebner basis for the Ptolemy variety, and then uses (1.4) to compute the complex volumes. We have run the algorithm on all manifolds in the census of cusped hyperbolic manifolds with up to 8 simplices [9], all link complements with up to 11 crossings [9], as well as selected manifolds in the Regina census of closed, prime 3-manifolds with up to 11 simplices [4]. For $n = 2$, the Groebner basis computations terminate for all but a few manifolds. In particular, they terminate for all census manifolds with ≤ 7 simplices and all link complements with ≤ 15 simplices and all but 3 link complements with 16 simplices. For $n = 3$, computations are feasible with up to 4 simplices, but for $n = 4$ the computations run out of memory for all manifolds with more than 2 simplices. It would be interesting to perform numerical calculations for $n \geq 4$. Our computations have revealed numerous (numerical) examples of linear combinations as predicted by Neumann's conjecture. To the best of our knowledge, our examples are the first concrete computations of the Cheeger-Chern-Simons invariant (complex volume) for $n > 2$.

1.1. Statement of our results. This section gives a brief summary of our main results. More details can be found in the paper.

Let M be a compact, oriented 3-manifold with (possibly empty) boundary, and let K be a closed 3-cycle (triangulated complex; see Definition 4.1) homeomorphic to the space obtained from M by collapsing each boundary component to a point. We identify each of the simplices of K with a standard simplex

$$(1.6) \quad \Delta_n^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid 0 \leq x_i \leq n, \quad x_0 + x_1 + x_2 + x_3 = n\}.$$

For a non-negative integer l , let $T^3(l)$ be the set of tuples (a_0, a_1, a_2, a_3) of non-negative integers with sum l . Note that $T^3(n)$ is the set of integral points of Δ_n^3 . We shall primarily be interested in the set $\mathring{T}^3(n)$ of non-vertex integral points of Δ_n^3 , i.e. the subset of $T^3(n)$ of tuples with all coordinates less than n .

Definition 1.1. A Ptolemy cochain on Δ_n^3 is an assignment $\mathring{T}^3(n) \rightarrow \mathbb{C}^*$, $t \mapsto c_t$, of a non-zero complex number c_t to each (non-vertex) integral point t of Δ_n^3 such that for each $\alpha \in T^3(n-2)$, the *Ptolemy relation*

$$(1.7) \quad c_{\alpha_{03}} c_{\alpha_{12}} + c_{\alpha_{01}} c_{\alpha_{23}} = c_{\alpha_{02}} c_{\alpha_{13}}$$

is satisfied. Here, α_{ij} denotes the integral point $\alpha + e_i + e_j$. A Ptolemy cochain on K is a Ptolemy cochain c^i on each simplex Δ_i of K such that the Ptolemy coordinates agree on identified faces.

Remark 1.2. The name is inspired by the resemblance of (1.7) with the Ptolemy relation between the lengths of the sides and diagonals of an inscribed quadrilateral (see Figure 1). In the work of Fock and Goncharov [13], the Ptolemy relations appear as relations between coordinates on the higher Teichmüller space when the triangulation of a surface is changed by a flip.

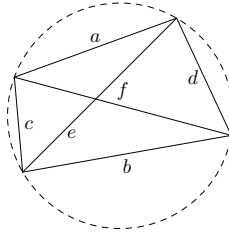


FIGURE 1. A quadrilateral is inscribed in a circle if and only if $ab + cd = ef$.

It follows immediately from the definition that the set of Ptolemy cochains on K is an algebraic set $P_n(K)$, which we shall refer to as the *Ptolemy variety*. As we shall see, the Ptolemy variety parametrizes *generically decorated* boundary-unipotent representations on K in the sense of Definition 5.2. If $\rho: \pi_1(M) \rightarrow \mathrm{SL}(n, \mathbb{C})$ is a boundary-unipotent representation, and E_ρ is the corresponding flat $\mathrm{SL}(n, \mathbb{C})$ -bundle, a decoration of ρ determines a reduction of $(E_\rho)_{\partial M}$ to an N -bundle (see Proposition 4.6).

The extended pre-Bloch group $\widehat{\mathcal{P}}(\mathbb{C})$ is generated by tuples $(u, v) \in \mathbb{C}^2$ with $e^u + e^v = 1$. We refer to Section 3 for a review. Using (1.7), we obtain that a Ptolemy cochain c on Δ_n^3 gives rise to an element

$$(1.8) \quad \lambda(c) = \sum_{\alpha \in T^3(n-2)} (\tilde{c}_{\alpha_{03}} + \tilde{c}_{\alpha_{12}} - \tilde{c}_{\alpha_{02}} - \tilde{c}_{\alpha_{13}}, \tilde{c}_{\alpha_{01}} + \tilde{c}_{\alpha_{23}} - \tilde{c}_{\alpha_{02}} - \tilde{c}_{\alpha_{13}}) \in \widehat{\mathcal{P}}(\mathbb{C}),$$

where the tilde denotes a branch of logarithm (the particular choice is inessential). We thus have a map

$$(1.9) \quad \lambda: P_n(K) \rightarrow \widehat{\mathcal{P}}(\mathbb{C}), \quad c \mapsto \sum_i \epsilon_i \lambda(c^i),$$

where the sum is over the simplices of K . Let $R_{\mathrm{SL}(n, \mathbb{C}), N}(M)/\mathrm{Conj}$ denote the set of conjugacy classes of boundary-unipotent representations $\pi_1(M) \rightarrow \mathrm{SL}(n, \mathbb{C})$.

Theorem 1.3 (Proof in Section 9.2). *A Ptolemy cochain c uniquely determines a boundary-unipotent representation $\rho(c) \in R_{\mathrm{SL}(n, \mathbb{C}), N}(M)/\mathrm{Conj}$. The map λ has image in the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$, and we have a commutative diagram.*

$$(1.10) \quad \begin{array}{ccc} P_n(K) & \xrightarrow{\lambda} & \widehat{\mathcal{B}}(\mathbb{C}) \\ \downarrow \rho & & \downarrow R \\ R_{\mathrm{SL}(n, \mathbb{C}), N}(M)/\mathrm{Conj} & \xrightarrow{i \mathrm{Vol}_{\mathbb{C}}} & \mathbb{C}/4\pi^2\mathbb{Z} \end{array}$$

If $c \in P_n(K)$ is a Ptolemy cochain on K , $\lambda(c)$ only depends on the representation $\rho(c)$. Moreover, if the triangulation is sufficiently fine (a single barycentric subdivision suffices), the map ρ is surjective. \square

The above theorem gives an efficient algorithm for computing the set of complex volumes. For numerous examples, see Section 10.

Corollary 1.4. A boundary-unipotent representation $\rho: \pi_1(M) \rightarrow \mathrm{SL}(n, \mathbb{C})$ determines an element $[\rho] \in \widehat{\mathcal{B}}(\mathbb{C})$ such that $R([\rho]) = i \mathrm{Vol}_{\mathbb{C}}(\rho)$. \square

The Cheeger-Chern-Simons invariant can be viewed as a characteristic class $H_3(\mathrm{SL}(n, \mathbb{C})) \rightarrow \mathbb{C}/4\pi^2\mathbb{Z}$, and the result underlying the proof of commutativity of (1.10) is Theorem 1.5 below, giving an explicit cocycle formula for the Cheeger-Chern-Simons class. The formula generalizes the formula in Goette-Zickert [14] for $n = 2$. Recall that a homology class can be represented by a formal sum of tuples (g_0, \dots, g_3) . To such a tuple, we can assign a Ptolemy cochain $c(g_0, \dots, g_3)$ defined by

$$(1.11) \quad c(g_0, \dots, g_3)_t = \det(\{g_0\}_{t_0} \cup \dots \cup \{g_3\}_{t_3}), \quad t = (t_0, \dots, t_3),$$

where $\{g_i\}_{t_i}$ denotes the ordered set consisting of the first t_i column vectors of g_i . One can always represent a homology class by tuples, such that all the determinants (1.11) are non-zero.

Theorem 1.5 (Proof in Section 8). *The Cheeger-Chern-Simons class \widehat{c} factors as*

$$(1.12) \quad H_3(\mathrm{SL}(n, \mathbb{C})) \xrightarrow{\lambda} \widehat{\mathcal{B}}(\mathbb{C}) \xrightarrow{R} \mathbb{C}/4\pi^2\mathbb{Z},$$

where λ is induced by the map taking a tuple (g_0, \dots, g_3) to $\lambda(c(g_0, \dots, g_3)) \in \widehat{\mathcal{P}}(\mathbb{C})$. \square

We stress that the variety $P_n(K)$ depends on the triangulation of K , and may be empty. If a representation ρ is in the image of $P_n(K) \rightarrow R_{\mathrm{SL}(n, \mathbb{C}), N}(M)/\mathrm{Conj}$, we say that $P_n(K)$ detects ρ .

Let $\phi_n: \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SL}(n, \mathbb{C})$ denote the canonical irreducible representation. Note that when n is odd ϕ_n factors through $\mathrm{PSL}(2, \mathbb{C})$.

Theorem 1.6 (Proof in Section 11.1). *Suppose $M = \mathbb{H}^3/\Gamma$ is an oriented, hyperbolic manifold with finite volume and geometric representation $\rho_{\mathrm{geo}}: \pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$. If the triangulation of K has no non-essential edges, and if n is odd, $P_n(K)$ is non-empty and detects $\phi_n \circ \rho_{\mathrm{geo}}$. \square*

When n is even, $\phi_n \circ \rho_{\text{geo}}$ is only a representation in $p\text{SL}(n, \mathbb{C}) = \text{SL}(n, \mathbb{C}) / \langle \pm I \rangle$.

Definition 1.7. Let $\sigma \in Z^2(\Delta_n^3; \mathbb{Z}/2\mathbb{Z})$ be a cocycle. A $p\text{SL}(n, \mathbb{C})$ -Ptolemy cochain on Δ_n^3 with obstruction cocycle σ is an assignment of Ptolemy coordinates to the integral points of Δ_n^3 such that

$$(1.13) \quad \sigma_2 \sigma_3 c_{\alpha_{03}} c_{\alpha_{12}} + \sigma_0 \sigma_3 c_{\alpha_{01}} c_{\alpha_{23}} = c_{\alpha_{02}} c_{\alpha_{13}}.$$

Here $\sigma_i \in \mathbb{Z}/2\mathbb{Z} = \langle \pm 1 \rangle$ is the value of σ on the face opposite the i th vertex of Δ . A $p\text{SL}(n, \mathbb{C})$ -Ptolemy cochain on K with obstruction cocycle $\sigma \in Z^2(K; \mathbb{Z}/2\mathbb{Z})$ is a collection of $p\text{SL}(n, \mathbb{C})$ -cochains c^i on Δ_i with obstruction class σ_{Δ_i} such that the Ptolemy coordinates agree on common faces.

The set of $p\text{SL}(n, \mathbb{C})$ -Ptolemy cochains on K with obstruction cocycle σ is an algebraic set $P_n^\sigma(K)$, which up to canonical isomorphism, only depends on the cohomology class of σ . The obstruction class to lifting a boundary-unipotent representation in $p\text{SL}(n, \mathbb{C})$ to a boundary-unipotent representation in $\text{SL}(n, \mathbb{C})$ is a class in $H^2(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = H^2(K; \mathbb{Z}/2\mathbb{Z})$. For $\sigma \in H^2(K; \mathbb{Z}/2\mathbb{Z})$, let $R_{p\text{SL}(n, \mathbb{C}), N}^\sigma(M)$ denote the set of boundary-unipotent representations in $p\text{SL}(n, \mathbb{C})$ with obstruction class σ .

Theorem 1.8 (Proof in Section 9.2). *Let n be even. For each $\sigma \in H^2(K; \mathbb{Z}/2\mathbb{Z})$, we have a commutative diagram*

$$(1.14) \quad \begin{array}{ccc} P_n^\sigma(K) & \xrightarrow{\lambda} & \widehat{\mathcal{B}}(\mathbb{C})_{\text{PSL}} \\ \downarrow \rho & & \downarrow R \\ R_{p\text{SL}(n, \mathbb{C}), N}^\sigma(M) / \text{Conj} & \xrightarrow{i \text{Vol}_{\mathbb{C}}} & \mathbb{C} / \pi^2 \mathbb{Z} \end{array}$$

The extended Bloch group element of a Ptolemy cochain c only depends on the representation $\rho(c)$, and if the triangulation of K is sufficiently fine, ρ is surjective. If $M = \mathbb{H}^3/\Gamma$ is hyperbolic, and if K has no non-essential edges, $P_n^{\sigma_{\text{geo}}}(K)$ detects $\phi_n \circ \rho_{\text{geo}}$. Here σ_{geo} is the obstruction class to lifting the geometric representation to a boundary-unipotent representation in $\text{SL}(2, \mathbb{C})$. \square

Remark 1.9. In particular, a boundary-unipotent representation in $p\text{SL}(n, \mathbb{C})$ determines an element in $\widehat{\mathcal{B}}(\mathbb{C})$ computing the complex volume. For $n = 2$, this (and also Corollary 1.4), was proved in Zickert [28].

Theorem 1.10 (Proof in Section 11). *Let ρ be a representation in $\text{SL}(2, \mathbb{C})$ or $\text{PSL}(2, \mathbb{C})$. The extended Bloch group element of $\phi_n \circ \rho$ is $\binom{n+1}{3}$ times that of ρ . In particular, composition with ϕ_n multiplies complex volume by $\binom{n+1}{3}$.* \square

When $n = 2$, Thurston's gluing equation variety $V(K)$ is another variety, which is often used to compute volume. It is given by an equation for each edge of K and an equation for each generator of the fundamental groups of the boundary-components of M (see Section 12).

Theorem 1.11 (Proof in Section 12). *Suppose M has h boundary components. There is a surjective regular map*

$$(1.15) \quad \coprod_{\sigma \in H^2(K; \mathbb{Z}/2\mathbb{Z})} P_2^\sigma(K) \rightarrow V(K)$$

with fibers $(\mathbb{C}^*)^h$. \square

Remark 1.12. The Ptolemy variety offers significant computational advantage over the gluing equations. Other programs solve the gluing equations numerically, whereas the Ptolemy variety allows exact computations that are practical even if the manifold has many simplices.

As shown in Zickert [27], the extended Bloch group can also be defined over a number field F , and we have a canonical isomorphism $\widehat{\mathcal{B}}(F) \cong K_3^{\mathrm{ind}}(F)$.

Theorem 1.13 (Proof in Section 13). *Let F be a number field. A boundary-unipotent representation $\rho: \pi_1(M) \rightarrow \mathrm{SL}(n, F)$ determines an element of $\widehat{\mathcal{B}}(F) = K_3^{\mathrm{ind}}(F)$ such that for each embedding $\tau: F \rightarrow \mathbb{C}$, we have*

$$(1.16) \quad R(\tau([\rho])) = i \mathrm{Vol}_{\mathbb{C}}(\tau \circ \rho).$$

If ρ is irreducible, $[\rho]$ lies in $\widehat{\mathcal{B}}(\mathrm{Tr}(\rho))$, where $\mathrm{Tr}(\rho) \subset F$ is the trace field of ρ . \square

1.2. Neumann's conjecture. The fact that (1.9) has image in $\widehat{\mathcal{B}}(\mathbb{C})$ as opposed to $\widehat{\mathcal{P}}(\mathbb{C})$ has very interesting conjectural consequences. It is well known (see e.g. Suslin [24]) that the Bloch group $\mathcal{B}(\mathbb{C})$ is a \mathbb{Q} -vector space, and Walter Neumann has conjectured that it is generated by Bloch invariants of hyperbolic manifolds. More generally, Walter Neumann has proposed the following stronger conjecture [19]:

Conjecture 1.14. Let $F \subset \mathbb{C}$ be a concrete number field which is not in \mathbb{R} . The Bloch group $\mathcal{B}(F)$ is generated (integrally) modulo torsion by hyperbolic manifolds with invariant trace field contained in F .

Using Theorems 1.3 and 1.8, Conjecture 1.14 implies:

Conjecture 1.15. Let ρ be a boundary-unipotent representation of $\pi_1(M)$ in $\mathrm{SL}(n, \mathbb{C})$ or $p\mathrm{SL}(n, \mathbb{C})$. There exist hyperbolic 3-manifolds M_1, \dots, M_k and integers r_1, \dots, r_k such that

$$(1.17) \quad \mathrm{Vol}_{\mathbb{C}}(\rho) = \sum r_i \mathrm{Vol}_{\mathbb{C}}(M_i) \in \mathbb{C}/i\pi^2\mathbb{Q}.$$

In particular, $\mathrm{Vol}(\rho) = \sum r_i \mathrm{Vol}(M_i) \in \mathbb{R}$.

We give some examples in Section 10.

1.3. Overview of the paper. Section 2 gives a detailed review of the Cheeger-Chern-Simons classes for flat bundles. Many details are included in order to give a self-contained proof of (1.2). Section 3 gives a brief review of the two variants of the extended Bloch group, and Section 4 reviews the theory, introduced in Zickert [28], of decorated representations and relative fundamental classes. In Section 5, we introduce the notion of *generic* decorations and define the Ptolemy variety $P_n(K)$. In Section 6, we construct a chain complex of Ptolemy cochains, and use it to construct a map from $H_3(\mathrm{SL}(n, \mathbb{C}), N)$ to $\widehat{\mathcal{B}}(\mathbb{C})$ commuting with stabilization. This shows that a decorated boundary-unipotent representation determines an element in the extended Bloch group, which is given explicitly in terms of the Ptolemy coordinates. In Section 7, we show that the extended Bloch group element of a decorated representation is independent of the decoration, and in Section 8, we show that the Cheeger-Chern-Simons class is given as in Theorem 1.5. In Section 9, we show that the Ptolemy variety parametrizes generically decorated representations, and give an explicit formula for recovering a representation from its Ptolemy coordinates. In Section 10, we give some examples of computations, and list some interesting findings. Section 11 discusses the irreducible representations of $\mathrm{SL}(2, \mathbb{C})$, and Section 12 discusses the relationship to Thurston's gluing equations when $n = 2$. Finally, Section 13 is a brief discussion of other fields.

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2. THE CHEEGER-CHERN-SIMONS CLASSES

The Cheeger-Chern-Simons classes [7, 6] are characteristic classes of principal bundles with connection. For general bundles, the characteristic classes are differential characters [6], but for flat bundles they reduce to ordinary (singular) cohomology classes. In this paper we will focus exclusively on flat bundles. Let \mathbb{F} denote either \mathbb{R} or \mathbb{C} , and let Λ be a proper subring of \mathbb{F} . Let G be a Lie group over \mathbb{F} with finitely many components. There is a characteristic class $S_{P,u}$ for each pair (P, u) consisting of an invariant polynomial $P \in I^k(G; \mathbb{F})$ and a class $u \in H^{2k}(BG; \Lambda)$, whose image in $H^{2k}(BG; \mathbb{F})$ equals $W(P)$, where W is the Chern-Weil homomorphism

$$(2.1) \quad W: I^k(G; \mathbb{F}) \rightarrow H^{2k}(BG; \mathbb{F}).$$

The characteristic class $S_{P,u}$ associates to each flat G -bundle $E \rightarrow M$ a cohomology class $S_{P,u}(E) \in H^{2k-1}(M; \mathbb{F}/\Lambda)$.

2.1. Simply connected, simple Lie groups. If G is simply connected and simple, $H^1(G; \mathbb{Z})$ and $H^2(G; \mathbb{Z})$ are trivial, and $H^3(G; \mathbb{Z}) \cong \mathbb{Z}$. Hence, by the Serre spectral sequence for the universal bundle, we have an isomorphism

$$(2.2) \quad S: H^4(BG; \mathbb{Z}) \cong H^3(G; \mathbb{Z}) \cong \mathbb{Z}$$

called the *suspension*. The Killing form on G defines an invariant polynomial $B \in I^2(G; \mathbb{F})$, and since B is real on the maximal compact subgroup K of G , $W(B)$ is a real class. Hence, there exists a unique positive real number α such that $W(\alpha B)$ is a generator of $H^4(BG; 4\pi^2\mathbb{Z})$. We refer to αB as the *renormalized Killing form*, and denote the Cheeger-Chern-Simons class $S_{\alpha B, W(\alpha B)}$ by \hat{c} .

Recall that every class in $H^3(G; \mathbb{F})$ can be represented by a G -invariant 3-form. The following is well known (see e.g. Kamber-Tondeur [16, (5.74) p. 116]).

Proposition 2.1. Let $P \in I^2(G; \mathbb{F})$. The suspension of $W(P)$ is represented by the invariant 3-form

$$(2.3) \quad \sigma(P) = -\frac{1}{6}P(\omega \wedge [\omega, \omega]) \in \Omega^3(G; \mathbb{F})^G$$

where ω is the Maurer-Cartan form on G . □

Let $E \rightarrow M$ be a G -bundle with flat connection θ . We can view θ as a map $\mathfrak{g}^* \rightarrow \Omega^1(E; \mathbb{F})$, so by taking exterior powers, θ induces a map

$$(2.4) \quad \theta: \Omega^3(G)^G = \wedge^3(\mathfrak{g}^*) \rightarrow \Omega^3(E; \mathbb{F}).$$

Note that $\theta(\sigma(P)) = -\frac{1}{6}P(\theta \wedge [\theta, \theta])$. In the following, P denotes the renormalized Killing form.

Proposition 2.2 ([6, Proposition 2.8]). Let $E \rightarrow M$ be a G -bundle, with flat connection θ , over a closed 3-manifold M . The cohomology class $\hat{c}(E) \in H^3(M; \mathbb{F}/4\pi^2\mathbb{Z})$ satisfies

$$(2.5) \quad \hat{c}(E)([M]) = \int_M s^*(\theta(\sigma(P))) \in \mathbb{F}/4\pi^2\mathbb{Z},$$

where s is a section of E (which exists since G is 2-connected). □

Remark 2.3. Since $\sigma(P) \in H^3(G; 4\pi^2\mathbb{Z})$ is a generator, it follows that a change of section changes the integral by a multiple of $4\pi^2\mathbb{Z}$.

Example 2.4. For $G = \mathrm{SL}(n, \mathbb{C})$, the renormalized killing form P equals $\frac{1}{2}\mathrm{Tr}$, where Tr is the trace form $(A, B) \mapsto \mathrm{Tr}(AB)$. For a flat connection, $d\theta = -\frac{1}{2}[\theta, \theta] = -\theta \wedge \theta$, so (2.5) yields

$$(2.6) \quad \widehat{c}(E)([M]) = \frac{1}{2} \int_M s^* \left(\mathrm{Tr}(\theta \wedge d\theta + \frac{2}{3} \theta \wedge \theta \wedge \theta) \right) \in \mathbb{C}/4\pi^2\mathbb{Z}$$

recovering the Chern-Simons integral (1.1). Note that P also equals the (renormalized) second Chern-polynomial c_2 . It thus follows that $\widehat{c} = \widehat{c}_2$.

2.2. Complex groups and volume. Recall that there is a 1-1 correspondence between flat G -bundles over M and representations $\pi_1(M) \rightarrow G$ up to conjugation. This correspondence takes a flat bundle to its holonomy representation. If $\rho: \pi_1(M) \rightarrow G$ is a representation, we let E_ρ denote the corresponding flat bundle. In the following G denotes a simply connected, simple, complex Lie group, and M a closed, oriented 3-manifold. The following definition is motivated by Theorem 2.8 below.

Definition 2.5. The *complex volume* $\mathrm{Vol}_{\mathbb{C}}(\rho)$ of a representation $\rho: \pi_1(M) \rightarrow G$ is defined by

$$(2.7) \quad \widehat{c}(E_\rho)(M) = i \mathrm{Vol}_{\mathbb{C}}(\rho) \in \mathbb{C}/4\pi^2\mathbb{Z}.$$

The *volume* $\mathrm{Vol}(\rho)$ of ρ is the real part of $\mathrm{Vol}_{\mathbb{C}}(\rho)$.

The bundle E_ρ is isomorphic to $\widetilde{M} \times_\rho G$, and we thus have a 1-1 correspondence between sections of E_ρ and ρ -equivariant maps $\widetilde{M} \rightarrow G$ such that $f: \widetilde{M} \rightarrow G$ corresponds to the section $s(x) = [\widetilde{x}, f(\widetilde{x})]$.

Lemma 2.6. For any ρ -equivariant map $f: \widetilde{M} \rightarrow G$, we have $i \mathrm{Vol}_{\mathbb{C}}(\rho) = \int_D f^*(\sigma(P))$, where D is a fundamental domain for M in \widetilde{M} .

Proof. For any invariant form $\eta \in \Omega^3(G)^G$, the form $\theta(\eta) \in \Omega^3(E_\rho; \mathbb{F})$ is induced by the pullback of η under the projection $\widetilde{M} \times G \rightarrow G$. Letting $\eta = \sigma(P)$, the result follows from (2.5). \square

Let $\mathbb{H}^3 = \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$ be hyperbolic 3-space. We identify the orthonormal frame bundle $F(\mathbb{H}^3)$ of \mathbb{H}^3 with $\mathrm{PSL}(2, \mathbb{C})$.

Lemma 2.7. For $G = \mathrm{SL}(2, \mathbb{C})$, $\sigma(P) = -h^* \wedge e^* \wedge f^*$, where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are the standard generators of $\mathfrak{sl}(2, \mathbb{C})$ over \mathbb{C} .

Proof. As in Example 2.4, $P = \frac{1}{2}\mathrm{Tr}$. Using the fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, it follows from (2.3) that $\sigma(P) \in \Omega^3(G)^G = \wedge^3(\mathfrak{g}^*)$ is given by

$$(2.8) \quad \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}, \quad (A, B, C) \mapsto -\frac{1}{2}\mathrm{Tr}(A[B, C]).$$

A simple computation shows that if $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}$ and $C = \begin{pmatrix} i & j \\ k & -i \end{pmatrix}$

$$(2.9) \quad -\frac{1}{2}\mathrm{Tr}(A[B, C]) = -\det \begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix} = -h^* \wedge e^* \wedge f^*(A, B, C).$$

This proves the result. \square

Theorem 2.8. Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold, and let $\rho: \pi_1(M) \rightarrow \mathrm{SL}(2, \mathbb{C})$ be a lift of the geometric representation. We have

$$(2.10) \quad \widehat{c}(E_\rho)([M]) = i(\mathrm{Vol}(M) + i\mathrm{CS}(M)) \text{ in } \mathbb{C}/2\pi^2\mathbb{Z},$$

where $\mathrm{CS}(M) = 2\pi^2 \mathrm{cs}(M)$, and $\mathrm{cs}(M)$ is the (Riemannian) Chern-Simons invariant [7, (6.2)].

Proof. The fact that the imaginary part equals volume is well known, and follows from the fact (see Dupont [11]) that the imaginary part of $\sigma(P)$ is cohomologous to the pullback of the hyperbolic volume form. Yoshida [26, Lemma 3.1] shows that the real part of the form $h^* \wedge e^* \wedge f^*$ equals $2\pi^2$ cs, where cs is the Riemannian Chern-Simons form on $F(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbb{C})$ (pulled back to $\mathrm{SL}(2, \mathbb{C})$). Note that the Riemannian connection on $F(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbb{C})$ descends to the Riemannian connection on $F(M) = \mathrm{PSL}(2, \mathbb{C})/\Gamma$. If $f: \widetilde{M} \rightarrow \mathrm{SL}(2, \mathbb{C})$ is ρ -equivariant, the composition

$$(2.11) \quad \widetilde{M} \xrightarrow{f} \mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{PSL}(2, \mathbb{C}) \longrightarrow \mathrm{PSL}(2, \mathbb{C})/\Gamma = F(M)$$

is ρ -invariant, and thus descends to a section of $F(M)$. The result now follows from Yoshida's result together with Lemma 2.7 and Lemma 2.6. \square

Remark 2.9. Note that Theorem 2.8 implies that modulo $2\pi^2$, the complex volume of a representation lifting the geometric representation only depends on M and not on the choice of lift.

Remark 2.10. Since P is real on K , the imaginary part of $\sigma(P)$ is cohomologous to an invariant 3-form on G/K . Since $H^3(\mathfrak{g}, \mathfrak{k}; \mathbb{R}) = \mathbb{R}$, there is a unique such form up to scaling. We may thus think of $\mathrm{Im}(\sigma(P))$ as a volume form.

2.3. The universal classes and group cohomology. The Cheeger-Chern-Simons classes are also defined for the universal flat bundle $EG^\delta \rightarrow BG^\delta$. For an explicit construction, we refer to Dupont-Kamber [12] or Dupont-Hain-Zucker [10]. In particular, we have a class $\widehat{c} \in H^3(BG^\delta; \mathbb{C}/4\pi^2\mathbb{Z})$. If $\rho: \pi_1(M) \rightarrow G$ is a representation, with classifying map $B\rho: M \rightarrow BG^\delta$, we thus have

$$(2.12) \quad \widehat{c}(B\rho_*([M])) = i \mathrm{Vol}_{\mathbb{C}}(\rho).$$

It is well known that the homology of BG^δ is the homology of the chain complex $C_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, where C_* is any free $\mathbb{Z}[G]$ -resolution of \mathbb{Z} . A convenient choice of free resolution is the complex C_* , generated in degree n by tuples (g_0, \dots, g_n) , and with boundary map given by

$$(2.13) \quad \partial(g_0, \dots, g_n) = \sum (-1)^i (g_0, \dots, \widehat{g}_i, \dots, g_n).$$

The homology of $C_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is denoted $H_*(G)$, so $H_*(G) = H_*(BG^\delta)$. Theorem 1.5 gives a concrete cocycle formula for $\widehat{c}: H_3(\mathrm{SL}(n, \mathbb{C})) \rightarrow \mathbb{C}/4\pi^2\mathbb{Z}$.

2.4. Compact manifolds with boundary. In Section 6.1 below, we construct a natural extension of $\widehat{c}: H_3(\mathrm{SL}(n, \mathbb{C})) \rightarrow \mathbb{C}/4\pi^2\mathbb{Z}$ to a homomorphism

$$(2.14) \quad \widehat{c}: H_3(\mathrm{SL}(n, \mathbb{C}), N) \rightarrow \mathbb{C}/4\pi^2\mathbb{Z},$$

where N is the subgroup of upper triangular matrices with 1's on the diagonal.

Definition 2.11. Let $\rho: \pi_1(M) \rightarrow \mathrm{SL}(n, \mathbb{C})$ be a boundary-unipotent representation. The *complex volume* of ρ is defined by

$$(2.15) \quad \widehat{c}(B\rho_*([M, \partial M])) = i \mathrm{Vol}_{\mathbb{C}}(\rho),$$

where $B\rho: (M, \partial M) \rightarrow (B\mathrm{SL}(n, \mathbb{C})^\delta, BN^\delta)$ is a classifying map for ρ .

Remark 2.12. Unlike when M is closed, the classifying map is not uniquely determined by ρ ; it depends on a choice of decoration (see Section 4). The complex volume, however, is independent of this choice.

2.5. Central elements of order 2. For any simple complex Lie group G , there is a canonical homomorphism (defined up to conjugation)

$$(2.16) \quad \phi_G: \mathrm{SL}(2, \mathbb{C}) \rightarrow G.$$

The element $s_G = \phi_G(-I)$ is a central element of G of order dividing 2, and equals $(-I)^{n+1}$ if $G = \mathrm{SL}(n, \mathbb{C})$ (see e.g. Fock-Goncharov [13, Corollary 2.1]). Let

$$(2.17) \quad pG = G / \langle s_G \rangle.$$

Note that ϕ_G descends to a homomorphism $\mathrm{PSL}(2, \mathbb{C}) \rightarrow pG$. The following follows easily from the Serre spectral sequence.

Proposition 2.13. Suppose s_G has order 2. The canonical map $p^*: H^4(BpG; \mathbb{Z}) \rightarrow H^4(BG; \mathbb{Z})$ is surjective with kernel of order dividing 4. \square

Corollary 2.14. There is a canonical characteristic class $\hat{c}: H_3(pG) \rightarrow \mathbb{C}/\pi^2\mathbb{Z}$.

Proof. By Proposition 2.13, there exists $u \in H^4(BpG; \pi^2\mathbb{Z})$ such that $p^*(u) = W(P) \in H^4(BG; \pi^2\mathbb{Z})$. Define $\hat{c} = S_{P,u}$. \square

In Section 6.3, we construct a homomorphism

$$(2.18) \quad \hat{c}: H_3(p\mathrm{SL}(n, \mathbb{C}), N) \rightarrow \mathbb{C}/\pi^2\mathbb{Z},$$

which extends \hat{c} to a characteristic class of bundles with boundary-unipotent holonomy. The complex volume of a representation in $p\mathrm{SL}(n, \mathbb{C})$ is defined as in Definition 2.11.

3. THE EXTENDED BLOCH GROUP

We use the conventions of Zickert [27]; the original reference is Neumann [18].

Definition 3.1. The *pre-Bloch group* $\mathcal{P}(\mathbb{C})$ is the free abelian group on $\mathbb{C} \setminus \{0, 1\}$ modulo the *five term relation*

$$(3.1) \quad x - y + \frac{y}{x} - \frac{1 - x^{-1}}{1 - y^{-1}} + \frac{1 - x}{1 - y} = 0, \quad \text{for } x \neq y \in \mathbb{C} \setminus \{0, 1\}.$$

The *Bloch group* is the kernel of the map $\nu: \mathcal{P}(\mathbb{C}) \rightarrow \wedge^2(\mathbb{C}^*)$ taking z to $z \wedge (1 - z)$.

Definition 3.2. The *extended pre-Bloch group* $\hat{\mathcal{P}}(\mathbb{C})$ is the free abelian group on the set

$$(3.2) \quad \hat{\mathbb{C}} = \{(e, f) \in \mathbb{C}^2 \mid \exp(e) + \exp(f) = 1\}$$

modulo the *lifted five term relation*

$$(3.3) \quad (e_0, f_0) - (e_1, f_1) + (e_2, f_2) - (e_3, f_3) + (e_4, f_4) = 0$$

if the equations

$$(3.4) \quad \begin{aligned} e_2 &= e_1 - e_0, & e_3 &= e_1 - e_0 - f_1 + f_0, & f_3 &= f_2 - f_1 \\ e_4 &= f_0 - f_1, & f_4 &= f_2 - f_1 + e_0 \end{aligned}$$

are satisfied. The *extended Bloch group* is the kernel of the map $\hat{\nu}: \hat{\mathcal{P}}(\mathbb{C}) \rightarrow \wedge^2(\mathbb{C})$ taking (e, f) to $e \wedge f$.

An element $(e, f) \in \widehat{\mathbb{C}}$ with $\exp(e) = z$ is called a *flattening* with *cross-ratio* z . Letting $\mu_{\mathbb{C}}$ denote the roots of unity in \mathbb{C}^* , we have a commutative diagram.

$$(3.5) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mu_{\mathbb{C}} & \xrightarrow{2\log} & \mathbb{C}/4\pi i\mathbb{Z} & \longrightarrow & \mathbb{C}^*/\mu_{\mathbb{C}} \longrightarrow 0 \\ & & \downarrow \chi & & \downarrow \chi & & \downarrow \\ 0 & \longrightarrow & \widehat{\mathcal{B}}(\mathbb{C}) & \longrightarrow & \widehat{\mathcal{P}}(\mathbb{C}) & \xrightarrow{\widehat{\nu}} & \wedge^2(\mathbb{C}) \longrightarrow K_2(\mathbb{C}) \longrightarrow 0 \\ & & \downarrow \pi & & \downarrow \pi & & \downarrow \\ 0 & \longrightarrow & \mathcal{B}(\mathbb{C}) & \longrightarrow & \mathcal{P}(\mathbb{C}) & \xrightarrow{\nu} & \wedge^2(\mathbb{C}^*) \longrightarrow K_2(\mathbb{C}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The map π is induced by the map taking a flattening to its cross-ratio, and χ is the map taking $e \in \mathbb{C}/4\pi i\mathbb{Z}$ to $(e, f + 2\pi i) - (e, f)$, where $f \in \mathbb{C}$ is any element such that $(e, f) \in \widehat{\mathbb{C}}$.

3.1. The regulator. By fixing a branch of logarithm, we may write a flattening with cross-ratio z as $[z; p, q] = (\log(z) + p\pi i, \log(1 - z) + q\pi i)$, where $p, q \in \mathbb{Z}$ are *even* integers. There is a well defined *regulator map*

$$(3.6) \quad \begin{aligned} R: \widehat{\mathcal{P}}(\mathbb{C}) &\rightarrow \mathbb{C}/4\pi^2\mathbb{Z}, \\ [z; p, q] &\mapsto \text{Li}_2(z) + \frac{1}{2}(\log(z) + p\pi i)(\log(1 - z) - q\pi i) - \pi^2/6. \end{aligned}$$

3.2. The $\text{PSL}(2, \mathbb{C})$ -variant of the extended Bloch group. There is another variant of the extended Bloch group using flattenings $[z; p, q]$, where p and q are allowed to be odd. This group is defined as above using the set

$$(3.7) \quad \widehat{\mathbb{C}}_{\text{odd}} = \{(e, f) \in \mathbb{C}^2 \mid \pm \exp(e) \pm \exp(f) = 1\},$$

and fits in a diagram similar to (3.5). We use a subscript PSL to denote the variant allowing odd flattenings. We have an exact sequence

$$(3.8) \quad 0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \widehat{\mathcal{B}}(\mathbb{C}) \longrightarrow \widehat{\mathcal{B}}(\mathbb{C})_{\text{PSL}} \longrightarrow 0.$$

For odd flattenings, the regulator (3.6) is well defined modulo $\pi^2\mathbb{Z}$.

Theorem 3.3 (Neumann [18], Goette-Zickert [14]). *There are natural isomorphisms*

$$(3.9) \quad H_3(\text{PSL}(2, \mathbb{C})) \cong \widehat{\mathcal{B}}(\mathbb{C})_{\text{PSL}}, \quad H_3(\text{SL}(2, \mathbb{C})) \cong \widehat{\mathcal{B}}(\mathbb{C})$$

such that the Cheeger-Chern-Simons classes agree with the regulators. □

The following result is needed in Section 7. The first part is proved in Zickert [27, Lemma 3.16], and the second has a similar proof, which we leave to the reader.

Lemma 3.4. For $(e, f) \in \widehat{\mathbb{C}}$ and $p, q \in \mathbb{Z}$, we have

$$(3.10) \quad (e + 2\pi ip, f + 2\pi iq) - (e, f) = \chi(qe - pf + 2pq\pi i) \in \widehat{\mathcal{P}}(\mathbb{C}),$$

$$(3.11) \quad (e + \pi ip, f + \pi iq) - (e, f) = \chi(qe - pf + pq\pi i) \in \widehat{\mathcal{P}}(\mathbb{C})_{\text{PSL}}. \quad \square$$

3.3. Arbitrary fields. In Zickert [27], extended Bloch groups $\widehat{\mathcal{B}}_E(F)$ and $\widehat{\mathcal{B}}_E(F)_{\mathrm{PSL}}$ are defined for an arbitrary field F and an extension E of F^* by \mathbb{Z} . The definitions are as above using the sets

$$(3.12) \quad \widehat{E}_F = \{(e, f) \in E^2 \mid \pi(e) + \pi(f) = 1\}, \quad (\widehat{E}_F)_{\mathrm{odd}} = \{(e, f) \in E^2 \mid \pm\pi(e) \pm \pi(f) = 1\}.$$

If F is a number field, the extended Bloch groups are up to canonical isomorphism independent of the choice of extension, so we may omit the subscript E .

Theorem 3.5 (Zickert [27, Theorem 1.1]). *Let F be a number field. There is a natural isomorphism*

$$(3.13) \quad K_3^{\mathrm{ind}}(F) \cong \widehat{\mathcal{B}}(F)$$

respecting Galois actions. □

Corollary 3.6 (Zickert [27, Corollary 7.14]). For each embedding $\tau: F \rightarrow \mathbb{C}$, the induced map $\tau: \widehat{\mathcal{B}}(F) \rightarrow \widehat{\mathcal{B}}(\mathbb{C})$ is injective. □

Corollary 3.7 (Galois descent; Zickert [27, Corollary 7.15]). Let $F_2 : F_1$ be a field extension. An element in $\widehat{\mathcal{B}}(F_2)$ is in $\widehat{\mathcal{B}}(F_1)$ if and only if it is invariant under all automorphisms of F_2 over F_1 . □

4. DECORATIONS OF REPRESENTATIONS

In this section we review the notion of decorated representations introduced in Zickert [28]. Throughout the section, G denotes an arbitrary group, not necessarily a Lie group. Let H be subgroup of G . An *ordered simplex* is a simplex with a fixed vertex ordering.

Definition 4.1. A *closed 3-cycle* is a cell complex K obtained from a finite collection of ordered 3-simplices Δ_i by gluing together pairs of faces using order preserving simplicial attaching maps. We assume that all faces have been glued, and that the space $M(K)$, obtained by truncating the Δ_i 's before gluing, is an oriented 3-manifold with boundary. Let ϵ_i be a sign indicating whether or not the orientation of Δ_i given by the vertex ordering agrees with the orientation of $M(K)$.

Note that up to removing disjoint balls (which does not effect the fundamental group), the manifold $M(K)$ only depends on the underlying topological space of K , and not on the choice of 3-cycle structure. Also note that for any compact, oriented 3-manifold M with (possibly empty) boundary, the space \widetilde{M} obtained from M by collapsing each boundary component to a point has a structure of a closed 3-cycle K such that $M = M(K)$.

Let K be a closed 3-cycle, and let $M = M(K)$. Let L denote the space obtained from the universal cover \widetilde{M} of M by collapsing each boundary component to a point. The 3-cycle structure of K induces a triangulation of L , and also a triangulation of M by truncated simplices. The covering map extends to a map $L \rightarrow K$, and the action of $\pi_1(M)$ on \widetilde{M} by deck transformations extends to an action on L , which is determined by fixing, once and for all, a base point in M together with one of its lifts. Note that the stabilizer of each zero cell is a *peripheral* subgroup of $\pi_1(M)$, i.e. a subgroup induced by inclusion of a boundary component.

Definition 4.2. Let H be a subgroup of G . A representation $\rho: \pi_1(M) \rightarrow G$ is a (G, H) -*representation* if the image of each peripheral subgroup is a conjugate of H .

Definition 4.3. Let ρ be a (G, H) -representation. A *decoration* (on K) of ρ is a ρ -equivariant assignment of a left H -coset to each vertex of L , i.e. if $\alpha \in \pi_1(M)$ and the coset $g_e H$ is assigned to e , the coset assigned to αe must be $\rho(\alpha)g_e H$.

Note that $g_e^{-1}\rho(\mathrm{Stab}(e))g_e \subset H$, where $\mathrm{Stab}(e)$ is the stabilizer of e .

Definition 4.4. Two decorations $\{g_e H\}$ and $\{g'_e H\}$ of ρ are *equivalent* if the corresponding subgroups $g_e^{-1}\rho(\text{Stab}(e))g_e$ and $g'_e{}^{-1}\rho(\text{Stab}(e))g'_e$ of H are conjugate (in H).

Note that if $\{g_e H\}$ is a decoration of ρ , then $\{gg_e H\}$ is a decoration of $g\rho g^{-1}$. Since we are only interested in representations up to conjugation, we consider such two decorations to be equal.

Remark 4.5. If we fix a lift $e_i \in L$ of each 0-cell of K , a decoration is uniquely determined by the cosets $g_i H$ assigned to e_i . Note that if $\{g_i H\}$ and $\{g'_i H\}$ are decorations, there exist $x_i \in N_G(H)/H$ such that $\{g'_i H\}$ is equivalent to $\{g_i x_i H\}$. If h is the number of boundary components of M , it thus follows that there is a transitive right action of the group $(N_G(H)/H)^h$ on the set of equivalence classes of decorations.

Proposition 4.6. Let E be a flat G -bundle over M whose holonomy representation is a (G, H) -representation ρ . There is a 1-1 correspondence between decorations of ρ up to equivalence, and reductions of $E_{\partial M}$ to an H -bundle over ∂M .

Proof. For each boundary component S_i of M , choose a base point in S_i and a path to the base point of M . This determines a lift e_i in L of the vertex of K corresponding to S_i , and an identification of $\pi_1(S_i)$ with $\text{Stab}(e_i) \subset \pi_1(M)$. If F is a reduction of $E_{\partial M}$, the holonomy representations $\rho_i: \pi_1(S_i) \rightarrow H$ of F_{S_i} are conjugate to ρ , so there exist $g_i \in G$ such that $g_i^{-1}\rho g_i = \rho_i$. Assigning the coset $g_i H$ to e_i yields a decoration, which up to equivalence is independent of the choice of g_i 's. On the other hand, a decoration assigns cosets $g_i H$ to e_i such that $g_i^{-1}\rho(\text{Stab}(e_i))g_i \subset H$. Hence, g_i defines an isomorphism of E_{S_i} with an H -bundle, which up to isomorphism only depends on the equivalence class of the decoration. \square

4.1. The fundamental class of a decorated representation. A flat G -bundle over M determines a classifying map $M \rightarrow BG^\delta$, where the δ indicates that G is regarded as a discrete group. It thus follows from Proposition 4.6 that a decorated representation $\rho: \pi_1(M) \rightarrow G$ determines a map

$$(4.1) \quad B\rho: (M, \partial M) \rightarrow (BG^\delta, BH^\delta).$$

In particular, ρ gives rise to a fundamental class

$$(4.2) \quad [\rho] = B\rho_*([M, \partial M]) \in H_3(G, H),$$

where, by definition, $H_*(G, H) = H_*(BG^\delta, BH^\delta)$. Note that the fundamental class is independent of the particular 3-cycle structure on K .

Recall that M is triangulated by truncated simplices. By restriction, a (G, H) cocycle on M determines a (G, H) -cocycle on each truncated simplex. Let $\overline{B}_*(G, H)$ denote the chain complex generated in degree n by (G, H) -cocycles on a truncated n -simplex. As proved in Zickert [28, Section 3], $\overline{B}_*(G, H)$ computes the homology groups $H_3(G, H)$. Note that a (G, H) -cocycle on M determines (up to conjugation) a decorated (G, H) -representation.

Proposition 4.7 (Zickert [28, Proposition 5.10]). Let τ be a (G, H) -cocycle on M representing a decorated (G, H) -representation ρ . The cycle

$$(4.3) \quad \sum \epsilon_i \tau_{\overline{\Delta}_i} \in \overline{B}_3(G, H),$$

represents the fundamental class of ρ . \square

5. GENERIC DECORATIONS AND PTOLEMY COORDINATES

In all of the following, $G = \mathrm{SL}(n, \mathbb{C})$, and N is the subgroup of upper triangular matrices with 1's on the diagonal. A (G, N) -representation $\rho: \pi_1(M) \rightarrow G$ is called *boundary-unipotent*. For a matrix $g \in G$ and a positive integer $i \leq n \in \mathbb{N}$, let $\{g\}_i$ be the ordered set consisting of the first i column vectors of g .

Definition 5.1. A tuple (g_0N, \dots, g_kN) of N -cosets is *generic* if for each tuple $t = (t_0, \dots, t_k)$ of non-negative integers with sum n , we have

$$(5.1) \quad c_t := \det \left(\bigcup_{i=0}^k \{g_i\}_{a_i} \right) \neq 0,$$

where the determinant is viewed as a function on ordered sets of n vectors in \mathbb{C}^n . The numbers c_t are called *Ptolemy coordinates*.

Definition 5.2. A decoration of a boundary-unipotent representation is *generic* if for each simplex Δ of L , the tuple of cosets assigned to the vertices of Δ is generic.

For a set X , let $C_*(X)$ be the acyclic chain complex generated in degree k by tuples (x_0, \dots, x_k) . If X is a G -set, the diagonal G -action makes $C_*(X)$ into a complex of $\mathbb{Z}[G]$ -modules. Let $C_*^{\mathrm{gen}}(G/N)$ be the subcomplex of $C_*(G/N)$ generated by generic tuples.

Proposition 5.3. The complex $C_*^{\mathrm{gen}}(G/N) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ computes the relative homology. If $\rho: \pi_1(M) \rightarrow G$ is a generically decorated representation, the fundamental class of ρ is represented by

$$(5.2) \quad \sum \epsilon_i (g_0^i N, g_1^i N, g_2^i N, g_3^i N) \in C_3^{\mathrm{gen}}(G/N),$$

where $(g_0^i N, \dots, g_3^i N)$ are the cosets assigned to lifts $\tilde{\Delta}_i$ of the Δ_i 's. \square

Proposition 5.3 is proved in Section 9. The idea is that a generic tuple canonically determines a (G, N) -cocycle on a truncated simplex. Hence, $C_*^{\mathrm{gen}}(G/N) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is isomorphic to a subcomplex of $\overline{B}_3(G, N)$, and the representation (5.2) of the fundamental class is then an immediate consequence of (4.3).

Proposition 5.4. After a single barycentric subdivision of K , every decoration of a boundary-unipotent representation $\rho: \pi_1(M) \rightarrow G$ is equivalent to a generic one.

Proof. After a barycentric subdivision of K , every simplex Δ of K has distinct vertices and at least three vertices of Δ are interior (link is a sphere). Fix lifts $e_i \in L$ of each interior vertex of K . Since the stabilizer of a lift of an interior vertex is trivial, assigning any coset $g_i H$ to e_i yields an equivalent decoration. Since the g_i 's can be chosen arbitrarily, the result follows. \square

5.1. The geometry of the Ptolemy coordinates. We canonically identify each ordered k -simplex with the standard simplex

$$(5.3) \quad \Delta_n^k = \left\{ (x_0, \dots, x_k) \in \mathbb{R}^{k+1} \mid 0 \leq x_i \leq n, \sum_{i=0}^k x_i = n \right\}.$$

Recall that a tuple (g_0N, \dots, g_kN) has a Ptolemy coordinate for each tuple of $k+1$ non-negative integers summing to n . In other words, there is a Ptolemy coordinate for each integral point of Δ_n^k .

Definition 5.5. A *Ptolemy cochain* on Δ_n^k is an assignment of a non-zero complex number c_t to each integral point t of Δ_n^k such that the c_t 's are the Ptolemy coordinates of some tuple $(g_0N, \dots, g_kN) \in C_k^{\text{gen}}(G/N)$. A Ptolemy cochain on K is a Ptolemy cochain on each simplex Δ_i of K such that the Ptolemy coordinates agree on identified faces.

Note that a generically decorated boundary-unipotent representation determines a Ptolemy cochain on K . In Section 9, we show that every Ptolemy cochain is induced by a unique decorated representation. For a natural number l , consider the set

$$(5.4) \quad T^k(l) = \{(a_0, \dots, a_k) \in \mathbb{Z}_{\geq 0}^k \mid \sum_{i=0}^n a_i = l\}.$$

Note that the set $T^k(n)$ parametrizes the set of integral points of Δ_n^k .

Lemma 5.6. The number of elements in $T^k(l)$ is $\binom{l+k}{k}$.

Proof. The map $(a_0, \dots, a_k) \mapsto \{a_0 + 1, a_0 + a_1 + 2, \dots, a_0 + \dots + a_{k-1} + k\}$ gives a bijection between $T^k(l)$ and subsets of $\{1, \dots, l+k\}$ with k elements. \square

Let e_i , $0 \leq i \leq k$, be the i th standard basis vector of \mathbb{Z}^{k+1} . For each $\alpha \in T^k(n-2)$, the points $\alpha + 2e_i$ in Δ_n^k span a simplex $\Delta^k(\alpha)$, whose integral points are the points $\alpha_{ij} := \alpha + e_i + e_j$, see Figure 2. We refer to $\Delta^k(\alpha)$ as a *subsimplex* of Δ_n^k . By Lemma 5.6, Δ_n^3 has $\binom{n+3}{3}$ integral points and $\binom{n+1}{3}$ subsimplices.

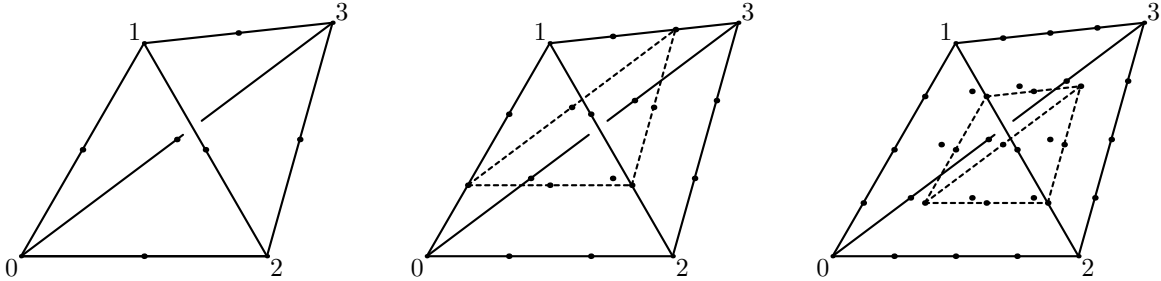


FIGURE 2. The integral points on Δ_n^3 for $n = 2, 3$ and 4 . The indicated subsimplices correspond to $\alpha = (0, 1, 0, 0)$ and $\alpha = (0, 1, 1, 0)$.

Proposition 5.7 (Fock-Goncharov [13, Lemma 10.3]). The Ptolemy coordinates of a generic tuple (g_0N, g_1N, g_2N, g_3N) satisfy the *Ptolemy relations*

$$(5.5) \quad c_{\alpha_{03}} c_{\alpha_{12}} + c_{\alpha_{01}} c_{\alpha_{23}} = c_{\alpha_{02}} c_{\alpha_{13}}, \quad \alpha \in T^3(n-2).$$

Proof. Let $\alpha = (a_0, a_1, a_2, a_3) \in T^3(n-2)$. By performing row operations, we may assume that the first $n-2$ rows of the $n \times (n-2)$ matrix

$$(5.6) \quad (\{g_0\}_{a_0}, \{g_1\}_{a_1}, \{g_2\}_{a_2}, \{g_3\}_{a_3})$$

are the standard basis vectors. Letting x_i and y_i denote the last two entries of $(g_i)_{a_i+1}$, the Ptolemy relation for α is then equivalent to the (Plücker) relation

$$(5.7) \quad \begin{pmatrix} x_0 & x_3 \\ y_0 & y_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} + \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} \begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix} = \begin{pmatrix} x_0 & x_2 \\ y_0 & y_2 \end{pmatrix} \begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix},$$

which is easily verified. \square

Note that the Ptolemy coordinate assigned to the i th vertex of Δ_n^k is $\det(\{g_i\}_n) = \det(g_i) = 1$. We shall thus often ignore the vertex points. Let $\mathring{T}^k(n)$ denote the non-vertex integral points of Δ_n^k . The following is proved in Section 9.

Proposition 5.8. For every assignment $c: \mathring{T}^3(n) \rightarrow \mathbb{C}^*$, $t \mapsto c_t$ satisfying the Ptolemy relations (5.5), there is a unique Ptolemy cochain on Δ_n^3 whose Ptolemy coordinates are c_t . \square

Corollary 5.9. The set of Ptolemy cochains on K is an algebraic set $P_n(K)$ called the *Ptolemy variety*. Its ideal is generated by the Ptolemy relations (5.5) (together with an extra equation making sure that all Ptolemy coordinates are non-zero). \square

Remark 5.10. It thus follows that Definition 5.5 agrees with Definition 1.1 when $k = 3$. When $k > 3$ and $n > 2$ there are further relations among the Ptolemy coordinates. We shall not need these here.

5.2. $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy coordinates. When n is even, a $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy cochain on Δ_n^k may be defined as in Definition 5.5. Note, however, that the Ptolemy coordinates are now only defined up to a sign. Since we are mostly interested in 3-cycles, the following definition is more useful.

Definition 5.11. Let $\Delta = \Delta_n^3$, and let $\sigma \in Z^2(\Delta; \mathbb{Z}/2\mathbb{Z})$ be a cellular 2-cocycle. A $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy cochain on Δ with *obstruction cocycle* σ is an assignment $c: \mathring{T}^3(n) \rightarrow \mathbb{C}^*$ satisfying the $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy relations

$$(5.8) \quad \sigma_2 \sigma_3 c_{\alpha_{03}} c_{\alpha_{12}} + \sigma_0 \sigma_3 c_{\alpha_{01}} c_{\alpha_{23}} = c_{\alpha_{02}} c_{\alpha_{13}}.$$

Here $\sigma_i \in \mathbb{Z}/2\mathbb{Z} = \langle \pm 1 \rangle$ is the value of σ on the face opposite the i th vertex of Δ . A $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy cochain on K with obstruction cocycle $\sigma \in Z^2(K; \mathbb{Z}/2\mathbb{Z})$ is a $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy-cochain c^i on each simplex Δ_i of K such that the Ptolemy coordinates agree on identified faces, and such that the obstruction cocycle of c^i is σ_{Δ_i} .

Note that for each $\sigma \in Z^2(K; \mathbb{Z}/2\mathbb{Z})$, the set of $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy-cochains on K form a variety $P_n^\sigma(K)$. We show in Section 9 that this variety only depends on the cohomology class of σ in $H^2(K; \mathbb{Z}/2\mathbb{Z}) = H^2(M, \partial M; \mathbb{Z}/2\mathbb{Z})$ and that the Ptolemy variety parametrizes generically decorated boundary-unipotent $p\mathrm{SL}(n, \mathbb{C})$ -representations whose obstruction class to lifting to a boundary-unipotent $\mathrm{SL}(n, \mathbb{C})$ -representation is σ . Note that when σ is the trivial cocycle taking all 2-cells to 1, $P^\sigma(K) = P(K)$.

5.3. Assigning cross-ratios and flattenings to subsimplices. For $x \in \mathbb{C} \setminus \{0\}$, let $\tilde{x} = \log(x)$, where \log is some fixed (set theoretic) section of the exponential map.

Given a Ptolemy cochain c on $\Delta_{n=2}^3$, we endow $\Delta_{n=2}^3$ with the shape of an ideal simplex with cross-ratio $z = \frac{c_{03}c_{12}}{c_{02}c_{13}}$ and a flattening

$$(5.9) \quad \lambda(c) = (\tilde{c}_{03} + \tilde{c}_{12} - \tilde{c}_{02} - \tilde{c}_{13}, \tilde{c}_{01} + \tilde{c}_{23} - \tilde{c}_{02} - \tilde{c}_{13}) \in \widehat{\mathcal{P}}(\mathbb{C}).$$

By Propositions 5.7 and 5.8, a Ptolemy cochain on Δ_n^3 induces a Ptolemy cochain c_α on each subsimplex $\Delta^3(\alpha)$. We thus have a map

$$(5.10) \quad \lambda: P_n(K) \rightarrow \widehat{\mathcal{P}}(\mathbb{C}), \quad c \mapsto \sum_i \epsilon_i \sum_{\alpha \in T^3(n-2)} \lambda(c_\alpha^i).$$

Similarly, we have a map $P_n^\sigma(K) \rightarrow \widehat{\mathcal{P}}(\mathbb{C})_{\mathrm{PSL}}$ defined by the same formula. We next prove that these maps have image in the respective extended Bloch groups.

6. A CHAIN COMPLEX OF PTOLEMY COCHAINS

Let Pt_k^n be the free abelian group on Ptolemy cochains on Δ_n^k . The usual boundary map induces a boundary map $Pt_k^n \rightarrow Pt_{k-1}^n$ and the natural map $C_*^{\text{gen}}(G/N) \rightarrow Pt_*$ taking a tuple (g_0N, \dots, g_kN) to its Ptolemy cochain is a chain map. The result below is proved in Section 9.

Proposition 6.1. A generic tuple is determined up to the diagonal G -action by its Ptolemy coordinates. \square

Corollary 6.2. The natural map induces an isomorphism

$$(6.1) \quad C_*^{\text{gen}}(G/N) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong Pt_*^n.$$

In particular, $H_*(G, N) = H_*(Pt_*^n)$. \square

Lemma 6.3. Let $c \in Pt_k^n$ be a Ptolemy cochain, and let $\alpha \in T^k(n-2)$. The Ptolemy coordinates $c_{\alpha_{ij}}$, $i \neq j$ are the Ptolemy coordinates of a unique Ptolemy cochain c_α on the subsimplex $\Delta^k(\alpha)$.

Proof. For $1 \leq k \leq 3$, this follows from Proposition 5.8. For $k > 3$, the result follows by induction, using the fact that 5 Ptolemy coordinates on Δ_2^3 determines the last. \square

A Ptolemy cochain c on Δ_n^k thus induces a Ptolemy cochain c_α on each subsimplex. We thus have maps

$$(6.2) \quad J_k^n: Pt_k^n \rightarrow Pt_k^2, \quad c \mapsto \sum_{\alpha \in T^k(n-2)} c_\alpha.$$

For a Ptolemy cochain $c \in Pt_k^n$ let $c_{\underline{i}} \in Pt_{k-1}^n$ be the induced Ptolemy cochain on the i th face of Δ_n^k , i.e. we have $\partial(c) = \sum_{i=0}^k (-1)^i c_{\underline{i}}$. Note that

$$(6.3) \quad (c_{\underline{i}})_{(a_0, \dots, a_{k-1})} = c_{(a_0, \dots, a_{i-1}, 0, a_i, \dots, a_{k-1})_{\underline{i}}} \in Pt_{k-1}^2.$$

For $\beta \in T^k(n-3)$, let $c_{\beta^i} = c_{(\beta + e_i)_{\underline{i}}} \in Pt_{k-1}^2$, and define $\partial_\beta(c) \in Pt_{k-1}^2$ by

$$(6.4) \quad \partial_\beta(c) = \sum_{i=0}^k (-1)^i c_{\beta^i} \in Pt_{k-1}^2.$$

The geometry is explained in Figure 3.

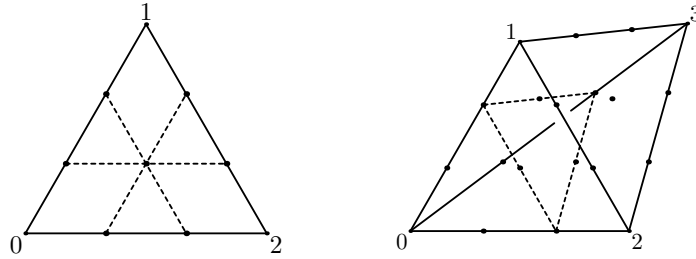


FIGURE 3. The dotted lines in the left figure indicate c_{β^0} , c_{β^1} and c_{β^2} for $n = 2$. The triangle in the right figure indicates c_{β^0} for $n = 3$.

Lemma 6.4. Let $c \in Pt_k^n$. We have

$$(6.5) \quad \partial(J_k^n(c)) - J_{k-1}^n(\partial(c)) = \sum_{\beta \in T^k(n-3)} \partial_\beta(c) \in Pt_{k-1}^2.$$

Proof. By (6.3), we have

$$\begin{aligned}
\partial(J_k^n(c)) - J_{k-1}^n(\partial(c)) &= \sum (-1)^i \sum_{\alpha \in T^k(n-2)} c_{\alpha_{\underline{i}}} - \sum (-1)^i \sum_{\substack{\alpha \in T^k(n-2) \\ a_i=0}} c_{\alpha_{\underline{i}}} \\
&= \sum (-1)^i \sum_{\substack{\alpha \in T^k(n-2) \\ a_i > 0}} c_{\alpha_{\underline{i}}} \\
&= \sum_{\beta \in T^k(n-3)} \sum (-1)^i c_{(\beta+e_i)_{\underline{i}}} \\
&= \sum_{\beta \in T^k(n-3)} \partial_{\beta}(c)
\end{aligned}$$

as desired \square

6.1. The map to the extended Bloch group. We wish to define a map

$$(6.6) \quad \lambda: H_3(\mathrm{SL}(n, \mathbb{C}), N) \rightarrow \widehat{\mathcal{B}}(\mathbb{C}).$$

Letting \tilde{x} denote a logarithm of x , we consider the maps

$$(6.7) \quad \lambda: Pt_3^2 \rightarrow \mathbb{Z}[\widehat{\mathbb{C}}], \quad c \mapsto (\tilde{c}_{03} + \tilde{c}_{12} - \tilde{c}_{02} - \tilde{c}_{13}, \tilde{c}_{01} + \tilde{c}_{23} - \tilde{c}_{02} - \tilde{c}_{13})$$

$$(6.8) \quad \mu: Pt_2^2 \rightarrow \wedge^2(\mathbb{C}), \quad c \mapsto -\tilde{c}_{01} \wedge \tilde{c}_{02} + \tilde{c}_{01} \wedge \tilde{c}_{12} - \tilde{c}_{02} \wedge \tilde{c}_{12} + \tilde{c}_{02} \wedge \tilde{c}_{02}.$$

Remark 6.5. The term $\tilde{c}_{02} \wedge \tilde{c}_{02}$ vanishes in $\wedge^2(\mathbb{C})$, but over general fields this term is needed. General fields are discussed in Section 13.

Lemma 6.6 (Zickert [27, Lemma 6.9]). Let $\widehat{\mathrm{FT}}$ be the subgroup of $\mathbb{Z}[\widehat{\mathbb{C}}]$ generated by the lifted five term relations. There is a commutative diagram

$$(6.9) \quad \begin{array}{ccccc} Pt_4^2 & \xrightarrow{\partial} & Pt_3^2 & \xrightarrow{\partial} & Pt_2^2 \\ \downarrow \lambda & & \downarrow \lambda & & \downarrow \mu \\ \mathbb{Z}[\widehat{\mathrm{FT}}] & \hookrightarrow & \mathbb{Z}[\widehat{\mathbb{C}}] & \xrightarrow{\hat{\nu}} & \wedge^2(\mathbb{C}). \end{array}$$

It follows that λ induces a map $\lambda: H_3(\mathrm{SL}(2, \mathbb{C}), N) \rightarrow \widehat{\mathcal{B}}(\mathbb{C})$. This map equals the map defined in Zickert [28, Section 7]. The fact that λ is independent of the choice of logarithm is proved in Zickert [28, Remark 6.11], and also follows from Proposition 7.7 below.

Lemma 6.7. For each $c \in Pt_4^n$ and each $\beta \in T^4(n-3)$, we have

$$(6.10) \quad \lambda(\partial_{\beta}(c)) = 0 \in \widehat{\mathcal{P}}(\mathbb{C}).$$

Proof. Let $(e_i, f_i) = \lambda(c_{\beta^i})$ be the flattening associated to c_{β^i} . We prove that the flattenings satisfy the five term relation by proving that the equations (3.4) are satisfied. We have

$$\begin{aligned}
e_0 &= \tilde{c}_{\beta+(1,1,0,0,1)} + \tilde{c}_{\beta+(1,0,1,1,0)} - \tilde{c}_{\beta+(1,1,0,1,0)} - \tilde{c}_{\beta+(1,0,1,0,1)} \\
e_1 &= \tilde{c}_{\beta+(1,1,0,0,1)} + \tilde{c}_{\beta+(0,1,1,1,0)} - \tilde{c}_{\beta+(1,1,0,1,0)} - \tilde{c}_{\beta+(0,1,1,0,1)} \\
e_2 &= \tilde{c}_{\beta+(1,0,1,0,1)} + \tilde{c}_{\beta+(0,1,1,1,0)} - \tilde{c}_{\beta+(1,0,1,1,0)} - \tilde{c}_{\beta+(0,1,1,0,1)}
\end{aligned}
\tag{6.11}$$

and it follows that $e_2 = e_1 - e_0$ as desired. The other 4 equations are proved similarly. \square

Lemma 6.8. For each $c \in Pt_3^n$ and each $\beta \in T^3(n-3)$, $\mu(\partial_{\beta}(c)) = 0 \in \wedge^2(\mathbb{C})$.

Proof. We have

$$(6.12) \quad \mu(c_{\beta^0}) = -\tilde{c}_{\beta+(1,1,1,0)} \wedge \tilde{c}_{\beta+(1,1,0,1)} + \tilde{c}_{\beta+(1,1,1,0)} \wedge \tilde{c}_{\beta+(1,0,1,1)} \\ - \tilde{c}_{\beta+(1,1,0,1)} \wedge \tilde{c}_{\beta+(1,0,1,1)} + \tilde{c}_{\beta+(1,1,0,1)} \wedge \tilde{c}_{\beta+(1,1,0,1)}.$$

Using this together with the similar formulas for $\mu(c_{\beta^i})$, we obtain that

$$\sum (-1)^i \mu(c_{\beta^i}) = 0 \in \wedge^2(\mathbb{C}),$$

proving the result. \square

Corollary 6.9. The map $\lambda \circ J_3^n$ induces a map

$$(6.13) \quad \lambda: H_3(\mathrm{SL}(n, \mathbb{C}), N) \rightarrow \hat{\mathcal{B}}(\mathbb{C}).$$

Proof. Using Lemma 6.4, this follows from Lemma 6.7 and Lemma 6.8. \square

Remark 6.10. For $n = 3$, this map agrees with the map considered in Zickert [27].

Definition 6.11. The *extended Bloch group element* of a decorated (G, N) -representation ρ is defined by $\lambda([\rho])$, where $[\rho] \in H_3(\mathrm{SL}(n, \mathbb{C}), N)$ is the fundamental class of ρ .

Note that if the decoration of ρ is generic, and c is the corresponding Ptolemy cochain, the extended Bloch group element is given by $\lambda(c)$, where $\lambda: P_n(K) \rightarrow \hat{\mathcal{P}}(\mathbb{C})$ is given by (5.10).

Proposition 6.12. The map $\lambda: P_n(K) \rightarrow \hat{\mathcal{P}}(\mathbb{C})$ has image in $\hat{\mathcal{B}}(\mathbb{C})$.

Proof. If $c \in P_n(K)$ is a Ptolemy cochain on K , we have a cycle $\alpha = \sum_i \epsilon_i c^i \in Pt_3^n$, and one easily checks that $\lambda(c)$ as defined in (5.10) equals $\lambda([\alpha])$. This proves the result. \square

6.2. Stabilization. We now prove that the map $\lambda: H_3(\mathrm{SL}(n, \mathbb{C}), N) \rightarrow \hat{\mathcal{B}}(\mathbb{C})$ respects stabilization. We regard $\mathrm{SL}(n-1, \mathbb{C})$ as a subgroup of $\mathrm{SL}(n, \mathbb{C})$ via the standard inclusion adding a 1 as the upper left entry.

Let $\pi: M(n, \mathbb{C}) \rightarrow M(n-1, \mathbb{C})$ be the map sending a matrix to the submatrix obtained by removing the first row and last column. The subgroup $D_k(\mathrm{SL}(n, \mathbb{C})/N)$ of $C_k^{\mathrm{gen}}(\mathrm{SL}(n, \mathbb{C})/N)$ generated by tuples (g_0N, \dots, g_kN) such that the upper left entry of each g_i is 1 and such that

$$(6.14) \quad (\pi(g_0)N, \dots, \pi(g_k)N) \in C_k^{\mathrm{gen}}(\mathrm{SL}(n-1, \mathbb{C})/N)$$

form an $\mathrm{SL}(n-1, \mathbb{C})$ -complex. Consider the $\mathrm{SL}(n-1, \mathbb{C})$ -invariant chain maps

$$(6.15) \quad \pi: D_*(\mathrm{SL}(n, \mathbb{C})/N) \rightarrow Pt_*^{n-1}$$

$$(6.16) \quad i: D_*(\mathrm{SL}(n, \mathbb{C})/N) \rightarrow Pt_*^n,$$

where the first map is induced by π and the second is induced by the inclusion $D_*(\mathrm{SL}(n, \mathbb{C})/N) \rightarrow C_*^{\mathrm{gen}}(\mathrm{SL}(n, \mathbb{C})/N)$. Let $D_k = D_k(\mathrm{SL}(n, \mathbb{C})/N) \otimes_{\mathbb{Z}[\mathrm{SL}(n-1, \mathbb{C})]} \mathbb{Z}$.

Lemma 6.13. The maps $\lambda \circ \pi$ and $\lambda \circ i$ from D_3 to $\hat{\mathcal{P}}(\mathbb{C})$ agree on cycles.

Proof. Let $c \in D_k$ be induced by a tuple $(g_0N, \dots, g_kN) \in D_k(\mathrm{SL}(n, \mathbb{C})/N)$, and let c^I be the collection of Ptolemy coordinates associated to (N, g_0N, \dots, g_kN) . Since some Ptolemy coordinates may be zero, c^I is not necessarily a Ptolemy cochain. Note, however, that c_α^I is a Ptolemy cochain for each $(a_0, \dots, a_{k+1}) \in T^{k+1}(n-2)$ with $a_0 = 0$. Note also that $c_\alpha^I \in Pt_{k+1}^2$ only depends on c . Hence, there is a map

$$(6.17) \quad P: D_k \rightarrow Pt_{k+1}^2, \quad c \mapsto \sum_{\substack{\alpha \in T^{k+1}(n-2) \\ a_0=0}} c_\alpha^I.$$

We wish to prove the following:

$$(6.18) \quad \partial P(c) + P\partial(c) = J_k^n(i(c)) - J_k^{n-1}(\pi(c)) + \sum_{\substack{\beta \in T^{k+1}(n-3) \\ b_0=0}} \partial_\beta(c^I) \in Pt_{k+1}^2.$$

Given this, the result follows immediately from Lemma 6.7.

One easily verifies that

$$(6.19) \quad c_{(\underline{1}, b_0, \dots, b_k)}^I = \pi(c)_{(b_0, \dots, b_k)} \in Pt_k^{n-1}, \quad (b_0, \dots, b_k) \in T^k(n-3).$$

$$(6.20) \quad c_{(\underline{0}, a_0, \dots, a_k)}^I = i(c)_{(a_0, \dots, a_k)}, \quad (a_0, \dots, a_k) \in T^k(n-2).$$

Using this, one has

$$\begin{aligned} \partial P(c) + P\partial(c) &= \sum_{\alpha \in T^k(n-2)} i(c)_\alpha + \sum_{i=1}^{k+1} (-1)^i \sum_{\substack{\alpha \in T^{k+1}(n-2) \\ a_0=0}} c_{\alpha_i}^I + \sum_{i=0}^k (-1)^i \sum_{\substack{\alpha \in T^{k+1}(n-2) \\ a_0=0, a_{i+1}=0}} c_{\alpha_{i+1}}^I \\ &= \sum_{\alpha \in T^k(n-2)} i(c)_\alpha + \sum_{i=1}^{k+1} (-1)^i \sum_{\substack{\alpha \in T^{k+1}(n-2) \\ a_0=0, a_i>0}} c_{\alpha_i}^I \\ &= \sum_{\alpha \in T^k(n-2)} i(c)_\alpha + \sum_{\substack{\beta \in T^{k+1}(n-3) \\ b_0=0}} \sum_{i=1}^{k+1} (-1)^i c_{\beta^i}^I \\ &= \sum_{\alpha \in T^k(n-2)} i(c)_\alpha - \sum_{\substack{\beta \in T^{k+1}(n-3) \\ b_0=0}} c_{\beta^0}^I + \sum_{\substack{\beta \in T^{k+1}(n-3) \\ b_0=0}} \partial_\beta(c^I) \\ &= J_k^n(i(c)) - J_k^{n-1}(\pi(c)) + \sum_{\substack{\beta \in T^{k+1}(n-3) \\ b_0=0}} \partial_\beta(c^I). \end{aligned}$$

This proves (6.18), hence the result. \square

Proposition 6.14. The map $\lambda: H_3(\mathrm{SL}(n, \mathbb{C}), N) \rightarrow \widehat{\mathcal{B}}(\mathbb{C})$ respects stabilization.

Proof. First note that π induces an isomorphism $D^0(\mathrm{SL}(n, \mathbb{C})/N) \cong C^0(\mathrm{SL}(n-1)/N)$. Using a standard cone argument, one easily checks that $D_*(\mathrm{SL}(n, \mathbb{C})/N)$ is a free $\mathrm{SL}(n-1, \mathbb{C})$ -resolution of $\mathrm{Ker}(D^0(\mathrm{SL}(n, \mathbb{C})/N) \rightarrow \mathbb{Z})$. Hence, D_* computes $H_*(\mathrm{SL}(n-1, \mathbb{C}), N)$, and the result follows from Lemma 6.13. \square

6.3. $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy cochains. When n is even, define pPt_*^n to be the complex of Ptolemy coordinates of generic tuples in $p\mathrm{SL}(n, \mathbb{C})/N$. The Ptolemy coordinates are defined as in (5.1), and take values in $\mathbb{C}^*/\langle \pm 1 \rangle$. As in (6.1), we have an isomorphism $C_*^{\mathrm{gen}}(p\mathrm{SL}(n, \mathbb{C})/N)_{p\mathrm{SL}(n, \mathbb{C})} \cong pPt_*^n$. For $c \in C^*/\langle \pm 1 \rangle$ let $\tilde{c} \in \mathbb{C}$ be the image of some fixed set theoretic section of $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \rightarrow \mathbb{C}^*/\langle \pm 1 \rangle$, e.g. $\frac{1}{2} \log(x^2)$ (the particular choice is inessential). The map

$$(6.21) \quad \lambda: pPt_3^2 \rightarrow \mathbb{Z}[\widehat{\mathbb{C}}_{\mathrm{odd}}], \quad c \mapsto (\tilde{c}_{03} + \tilde{c}_{12} - \tilde{c}_{02} - \tilde{c}_{13}, \tilde{c}_{01} + \tilde{c}_{23} - \tilde{c}_{02} - \tilde{c}_{13})$$

induces a map $H_3(\mathrm{PSL}(2, \mathbb{C}), N) \rightarrow \widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}$, which agrees with the map constructed in Zickert [28, Section 3]. By precomposing λ with the map $pJ_3^n: pPt_3^n \rightarrow pPt_3^2$ defined as in (6.2) we obtain a

map

$$(6.22) \quad \lambda: H_3(p\mathrm{SL}(n, \mathbb{C}), N) \rightarrow \widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}},$$

which commutes with stabilization. This proves that a decorated boundary-unipotent representation in $p\mathrm{SL}(n, \mathbb{C})$ determines an element in $\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}$. The proofs of the above assertions are word by word identical to their $\mathrm{SL}(n, \mathbb{C})$ -analogs.

7. INDEPENDENCE OF THE DECORATION

We now show that the extended Bloch group element of a decorated representation is independent of the decoration. We first prove that we can choose logarithms of the Ptolemy coordinates independently, without effecting the extended Bloch group element.

Definition 7.1. Let $c: \mathring{T}^k(n) \rightarrow \mathbb{C}^*$ be a Ptolemy cochain. A *lift* of c is an assignment $\tilde{c}: \mathring{T}^k(n) \rightarrow \mathbb{C}$ such that $\exp(\tilde{c}) = c$.

For any lift \tilde{c} of a Ptolemy cochain c on Δ_2^3 , we have a flattening

$$(7.1) \quad \lambda(\tilde{c}) = (\tilde{c}_{03} + \tilde{c}_{12} - \tilde{c}_{02} - \tilde{c}_{13}, \tilde{c}_{01} + \tilde{c}_{23} - \tilde{c}_{02} - \tilde{c}_{13}) \in \widehat{\mathbb{C}}.$$

Definition 7.2. The *log-parameters* of a flattening $(e, f) \in \widehat{\mathbb{C}}$ are defined by

$$(7.2) \quad w_{ij} = \begin{cases} e & \text{if } ij = 01 \text{ or } ij = 23 \\ -f & \text{if } ij = 12 \text{ or } ij = 03 \\ -e + f & \text{if } ij = 02 \text{ or } ij = 13. \end{cases}$$

Lemma 7.3. Let $\tilde{c}: T^3(2) \rightarrow \mathbb{C}$ be a lifted Ptolemy cochain, and let w_{ij} be the log-parameters of $\lambda(\tilde{c})$. Fix $i < j \in \{0, \dots, 3\}$ and let \tilde{c}' be the lifted Ptolemy cochain obtained from \tilde{c} by adding $2\pi\sqrt{-1}$ to \tilde{c}_{ij} . Then

$$(7.3) \quad \lambda(\tilde{c}') - \lambda(\tilde{c}) = \chi(w_{ij} + 2\pi\sqrt{-1}\delta_{ij}),$$

where δ_{ij} is 1 if $ij = 02$ or 13 and 0 otherwise.

Proof. Denote the flattening $\lambda(\tilde{c})$ by (e, f) . If $ij = 03$ or 12 , it follows from (7.1) that $\lambda(\tilde{c}') = (e + 2\pi\sqrt{-1}, f)$. Similarly, $\lambda(\tilde{c}') = (e, f + 2\pi\sqrt{-1})$ if $ij = 01$ or 23 , and $\lambda(\tilde{c}') = (e - 2\pi\sqrt{-1}, f - 2\pi\sqrt{-1})$ if $ij = 02$ or 13 . By Lemma 3.4,

$$(7.4) \quad \begin{aligned} (e + 2\pi\sqrt{-1}, f) - (e, f) &= \chi(-f) \\ (e, f + 2\pi\sqrt{-1}) - (e, f) &= \chi(e) \\ (e - 2\pi\sqrt{-1}, f - 2\pi\sqrt{-1}) - (e, f) &= \chi(-e + f + 2\pi\sqrt{-1}). \end{aligned}$$

This proves the result. \square

Let \tilde{c} be a lift of a Ptolemy cochain c . For each $\alpha \in T^3(n-2)$, \tilde{c} induces a lift \tilde{c}_α of c_α . Consider the element

$$(7.5) \quad \tau = \sum_{\alpha \in T^k(n-2)} \lambda(\tilde{c}_\alpha) \in \widehat{\mathcal{P}}(\mathbb{C}).$$

Fix a point $t_0 \in \mathring{T}^k(n)$. We wish to understand the effect on τ of adding $2\pi\sqrt{-1}$ to \tilde{c}_t . This changes τ into an element $\tau' \in \widehat{\mathcal{P}}(\mathbb{C})$. Let $w_{ij}(\alpha)$ denote the log-parameters of $\lambda(\tilde{c}_\alpha)$. Note that t_0 either lies on an edge, on a face, or in the interior of Δ_n^3 .

Lemma 7.4. Suppose t_0 is on the edge ij of Δ_n^3 . Then

$$(7.6) \quad \tau' - \tau = \chi(w_{ij}(\alpha) + 2\pi\sqrt{-1}\delta_{ij}),$$

where $\alpha = t - e_i - e_j$, (i.e. α is such that t_0 is an edge point of $\Delta^3(\alpha)$).

Proof. This follows immediately from Lemma 7.3. \square

Lemma 7.5. Suppose t_0 is on a face opposite vertex i . Then $\tau' - \tau = (-1)^i \chi(\kappa + 2\pi\sqrt{-1})$, where κ is given by

$$(7.7) \quad \kappa = \tilde{c}_{\eta_i(0,-1,1)} - \tilde{c}_{\eta_i(0,1,-1)} - (\tilde{c}_{\eta_i(-1,0,1)} - \tilde{c}_{\eta_i(1,0,-1)}) + \tilde{c}_{\eta_i(-1,1,0)} - \tilde{c}_{\eta_i(1,-1,0)},$$

where η_i inserts a zero as the i th vertex.

Proof. For simplicity assume $i = 0$. The other cases are proved similarly. There are exactly three α 's for which t_0 is an edge point of $\Delta^3(\alpha)$. These are

$$(7.8) \quad \alpha_0 = t_0 - (0, 0, 1, 1), \quad \alpha_1 = t_0 - (0, 1, 0, 1), \quad \alpha_2 = t_0 - (0, 1, 1, 0).$$

Note that $\tilde{c}_t = (\tilde{c}_{\alpha_0})_{23} = (\tilde{c}_{\alpha_1})_{13} = (\tilde{c}_{\alpha_2})_{12}$. Since adding $2\pi\sqrt{-1}$ to \tilde{c}_{t_0} leaves \tilde{c}_α unchanged unless $\alpha \in \{\alpha_0, \alpha_1, \alpha_2\}$, Lemma 7.3 implies that

$$(7.9) \quad \tau' - \tau = \chi(w_{23}(\alpha_0)) + \chi(w_{13}(\alpha_1) + 2\pi\sqrt{-1}) + \chi(w_{12}(\alpha_2)).$$

One easily checks that

$$(7.10) \quad \begin{aligned} w_{23}(\alpha_0) &= \tilde{c}_{(1,0,-1,0)} + \tilde{c}_{(0,1,0,-1)} - \tilde{c}_{(1,0,0,-1)} - \tilde{c}_{(0,1,-1,0)} \\ w_{13}(\alpha_1) &= \tilde{c}_{(1,0,0,-1)} + \tilde{c}_{(0,-1,1,0)} - \tilde{c}_{(1,-1,0,0)} - \tilde{c}_{(0,0,1,-1)} \\ w_{12}(\alpha_2) &= \tilde{c}_{(1,-1,0,0)} + \tilde{c}_{(0,0,-1,1)} - \tilde{c}_{(1,0,-1,0)} - \tilde{c}_{(0,-1,0,1)}, \end{aligned}$$

from which the result follows. \square

Lemma 7.6. If t_0 is an interior point, $\tau' = \tau$.

Proof. If t_0 is an interior point, there are six α 's for which t_0 is an edge point of $\Delta^3(\alpha)$. These are α_0, α_1 and α_2 as defined in (7.8) as well as

$$(7.11) \quad \alpha_3 = t_0 - (1, 1, 0, 0), \quad \alpha_4 = t_0 - (1, 0, 1, 0), \quad \alpha_5 = t_0 - (1, 0, 0, 1).$$

Again, by Lemma 7.3

$$(7.12) \quad \begin{aligned} \tau' - \tau &= \chi(w_{23}(\alpha_0)) + \chi(w_{13}(\alpha_1) + 2\pi\sqrt{-1}) + \chi(w_{12}(\alpha_2)) + \\ &\quad \chi(w_{01}(\alpha_3)) + \chi(w_{02}(\alpha_4) + 2\pi\sqrt{-1}) + \chi(w_{03}(\alpha_5)) \end{aligned}$$

Using (7.10) as well as

$$(7.13) \quad \begin{aligned} w_{01}(\alpha_3) &= \tilde{c}_{(0,-1,0,1)} + \tilde{c}_{(-1,0,1,0)} - \tilde{c}_{(0,-1,1,0)} - \tilde{c}_{(-1,0,0,1)} \\ w_{02}(\alpha_4) &= \tilde{c}_{(0,1,-1,0)} + \tilde{c}_{(-1,0,0,1)} - \tilde{c}_{(0,0,-1,1)} - \tilde{c}_{(-1,1,0,0)} \\ w_{03}(\alpha_5) &= \tilde{c}_{(0,0,1,-1)} + \tilde{c}_{(-1,1,0,0)} - \tilde{c}_{(0,1,0,-1)} - \tilde{c}_{(-1,0,1,0)} \end{aligned}$$

we see that all terms in (7.12) cancel out. Hence, $\tau' = \tau$. \square

Proposition 7.7. Let c be a Ptolemy cochain on K . For any lift \tilde{c} of c , the element

$$(7.14) \quad \lambda(\tilde{c}) = \sum_i \sum_{\alpha \in T^k(n-2)} \epsilon_i \lambda(\tilde{c}_\alpha^i) \in \hat{\mathcal{P}}(\mathbb{C})$$

is independent of the choice of lift. In particular, if c is the Ptolemy cochain of a decorated representation ρ , $\lambda(\tilde{c})$ is the extended Bloch group element of ρ .

Proof. Let \tilde{c} and \tilde{c}' be lifts of c . Let $t_0 \in T^3(n)$. We wish to prove that $\lambda(\tilde{c}) = \lambda(\tilde{c}')$. It is enough to prove this when \tilde{c}' is obtained from \tilde{c} by adding $2\pi\sqrt{-1}$ to \tilde{c}_t . If t_0 is an interior point, the result follows immediately from Lemma 7.6. If t_0 is a face point, t_0 lies in exactly two simplices of K , and it follows from Lemma 7.5 that the two contributions to the change in $\lambda(\tilde{c})$ appear with opposite signs (by (3.5), $2\chi(2\pi\sqrt{-1}) = 0$). Suppose t_0 is an edge point. Let C be the 3-cycle obtained by gluing together all the $\Delta^3(\alpha)$'s having t_0 as an edge point, using the face pairings induced from K . Let e be the (interior) 1-cell of C containing t_0 . The argument in Zickert [28, Theorem 6.5] shows that the total log-parameter around e is zero. It thus follows from Lemma 7.4 that adding $2\pi\sqrt{-1}$ to \tilde{c}_{t_0} changes $\lambda(\tilde{c})$ by 2-torsion which is trivial if and only if the number n of simplices in C for which t is a 02 edge or a 13 edge is even. Consider a curve λ in C encircling e . The vertex ordering induces an orientation on each face of each simplex of C , such that when λ passes through two faces of a simplex in C , the two orientations agree unless e is a 02 edge or a 13 edge. Since M is orientable, it follows that n is even. The second statement follows by letting $\tilde{c} = \log c$. \square

Proposition 7.8. The extended Bloch group element of a decorated boundary-unipotent representation is independent of the decoration.

Proof. By performing a barycentric subdivision if necessary, we may assume that any decoration is generic. Fix a lift $\tilde{\Delta}_i$ of each simplex Δ_i of K . A decoration D assigns a coset $g_j^i N$ to each vertex j of $\tilde{\Delta}_i$. Suppose we have another decoration D' of ρ with cosets $h_j^i N$. Since equivalent decorations have the same fundamental class, we may assume that $h_j^i = g_j^i d_j^i$, where the d_j^i 's are diagonal matrices (see Remark 4.5). Let c and c' denote the Ptolemy cochains on K induced by D and D' . Suppose $d_j^i = \text{diag}(d_{j0}^i, \dots, d_{j,n-1}^i)$. By (5.1) we have

$$(7.15) \quad c_t^{i'} = c_t^i \prod_{k=0}^{t_0} d_{0k}^i \prod_{k=0}^{t_1} d_{1k}^i \prod_{k=0}^{t_2} d_{2k}^i \prod_{k=0}^{t_3} d_{3k}^i.$$

Fix a lift \tilde{c} of c . Letting \log denote a logarithm, define a lift \tilde{c}' of c' by

$$(7.16) \quad \tilde{c}_t^{i'} = \tilde{c}_t^i + \sum_{k=0}^{t_0} \log(d_{0k}^i) + \sum_{k=0}^{t_1} \log(d_{1k}^i) + \sum_{k=0}^{t_2} \log(d_{2k}^i) + \sum_{k=0}^{t_3} \log(d_{3k}^i).$$

Using this, one easily checks that $\lambda(c_\alpha^i) = \lambda(c_\alpha^{i'})$ for each i and each $\alpha \in T^3(n)$. The result now follows from Proposition 7.7. \square

7.1. $p\text{SL}(n, \mathbb{C})$ -decorations. Let n be even. All results in this section have natural analogs for $p\text{SL}(n, \mathbb{C})$. The proofs of these are obtained by replacing $2\pi\sqrt{-1}$ by $\pi\sqrt{-1}$, and logarithms by lifts of $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*/\langle \pm 1 \rangle$. In particular, we have

Proposition 7.9. The fundamental class $[\rho] \in \widehat{\mathcal{B}}(\mathbb{C})_{\text{PSL}}$ of a decorated boundary-unipotent representation $\rho: \pi_1(M) \rightarrow p\text{SL}(n, \mathbb{C})$ is independent of the decoration. \square

8. A COCYCLE FORMULA FOR \widehat{c}

Let $i_*: H_3(\text{SL}(n, \mathbb{C})) \rightarrow H_3(\text{SL}(n, \mathbb{C}), N)$ denote the natural map. We wish to prove that the composition

$$(8.1) \quad H_3(\text{SL}(n, \mathbb{C})) \xrightarrow{i_*} H_3(\text{SL}(n, \mathbb{C}), N) \xrightarrow{\lambda} \widehat{\mathcal{B}}(\mathbb{C}) \xrightarrow{R} \mathbb{C}/4\pi^2\mathbb{Z}$$

equals the Cheeger-Chern-Simons class \widehat{c} . Note that i_* is induced by the map $(g_0, \dots, g_3) \mapsto (g_0N, \dots, g_3N)$.

We shall make use of the canonical isomorphisms

$$(8.2) \quad H_3(\mathrm{SL}(n, \mathbb{C})) \cong H_3(\mathrm{SL}(3, \mathbb{C})) \cong H_3(\mathrm{SL}(2, \mathbb{C})) \oplus K_3^M(\mathbb{C}).$$

The first isomorphism is induced by stabilization (see Suslin [23]) and the second isomorphism is the \pm -eigenspace decomposition with respect to the transpose-inverse involution on $\mathrm{SL}(3, \mathbb{C})$ (see Sah [21]).

Lemma 8.1 (Suslin [23]). Let $D \subset \mathrm{SL}(3, \mathbb{C})$ be the subgroup of diagonal matrices. The $K_3^M(\mathbb{C})$ summand of $H_3(\mathrm{SL}(3, \mathbb{C}))$ is generated by the elements $B\rho_*([T])$, where $T = S^1 \times S^1 \times S^1$ is the 3-torus, and $\rho: \pi_1(T) \rightarrow D$ is a representation. \square

Lemma 8.2. Let $T = S^1 \times S^1 \times S^1$ and let $\rho: \pi_1(T) \rightarrow D$ be a representation. The extended Bloch group element $[\rho] \in \widehat{\mathcal{B}}(\mathbb{C})$ of ρ is trivial.

Proof. We regard T as a cube C with opposite faces identified, and triangulate C as the cone on the triangulation on ∂C indicated in Figure 4 with cone point in the center. We order the vertices of each simplex by codimension, i.e. the 0-vertex is the cone point, the 1-vertex is a face point etc. Let $\rho: \pi_1(T) \rightarrow D$ be a representation, and pick a decoration of ρ by cosets in general position (the triangulation is such that this is always possible). Note that for every 3-simplex Δ of T , there is a unique opposite 3-simplex Δ^{opp} , such that the faces opposite the cone point are identified. We may assume that the cone point is decorated by the coset N . If a simplex Δ is decorated by the cosets (N, g_0N, g_1N, g_2N) , the simplex Δ^{opp} must be decorated by the cosets (N, dg_0N, dg_1N, dg_2N) , where d is the image of the generator of $\pi_1(T)$ pairing the faces of Δ and Δ^{opp} . It thus follows from (5.2) that the fundamental class is represented by a sum of terms of the form

$$(8.3) \quad (N, dg_0N, dg_1N, dg_2N) - (N, g_0N, g_1N, g_2N) \in C_3^{\mathrm{gen}}(\mathrm{SL}(n, \mathbb{C})/N).$$

Let c and c' be the Ptolemy cochains associated to the tuples (N, g_0N, g_1N, g_2N) and (N, dg_0N, dg_1N, dg_2N) . Letting $d = \mathrm{diag}(d_1, \dots, d_n)$, we have $c'_t = c_t \prod_{i=t_0}^n d_i$. Fix a lift \tilde{c} of c , and consider the lift

$$(8.4) \quad \tilde{c}'_t = \tilde{c}_t + \sum_{i=t_0}^n \log(d_i)$$

of c' . One now checks that $\lambda(\tilde{c}'_\alpha) = \lambda(\tilde{c}_\alpha)$ for all $\alpha \in \tilde{T}^k(n)$, so $\lambda(\tilde{c}) - \lambda(\tilde{c}') = 0$. This proves the result. \square

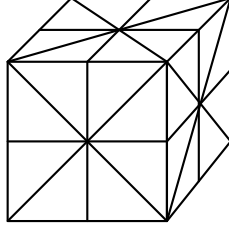
Theorem 8.3. *The composition $R \circ \lambda \circ i_*$ equals \widehat{c} .*

Proof. Since λ commutes with stabilization, it follows from Goette-Zickert [14] that $R \circ \lambda \circ i_* = \widehat{c}$ on $H_3(\mathrm{SL}(2, \mathbb{C}))$. Since \widehat{c} is zero on $K_3^M(\mathbb{C})$ (this follows from Lemma 8.1 and [6, Theorem 8.22]), the result follows from (8.2) and Lemma 8.2. \square

Remark 8.4. By defining $\widehat{c} = R \circ \lambda: H_3(\mathrm{SL}(n, \mathbb{C}), N) \rightarrow \mathbb{C}/4\pi^2\mathbb{Z}$, we have a natural extension of the Cheeger-Chern-Simons class to bundles with boundary-unipotent holonomy, and we can define the complex volume as in Definition 2.11.

9. RECOVERING A REPRESENTATION FROM ITS PTOLEMY COORDINATES

We now show that a Ptolemy cochain on K determines an explicit generically decorated boundary-unipotent representation. The idea is that a Ptolemy cochain canonically determines a (G, N) -cocycle on M .

FIGURE 4. Triangulation of ∂C

Definition 9.1. An $n \times n$ matrix A is *counter diagonal* if the only non-zero entries of A are on the lower left to upper right diagonal, i.e. $A_{ij} = 0$ unless $j = n - i + 1$. If $A_{ij} = 0$ for $j > n - i + 1$ (resp. $j < n - i + 1$), A is *upper* (resp. *lower*) *counter triangular*.

Given subsets I, J of $\{1, \dots, n\}$, let $A_{I,J}$ denote the submatrix of A whose rows and columns are indexed by I and J , respectively. If $|I| = |J|$, let $|A|_{I,J}$ denote the minor $\det(A_{I,J})$. Let I^c denote $\{1, \dots, n\} \setminus I$.

Recall that the adjugate $\text{Adj}(A)$ of a matrix A is the matrix whose ij th entry is $(-1)^{i+j}|A|_{\{j\}^c, \{i\}^c}$. It is well known that $\text{Adj}(A) = \det(A)A^{-1}$. The following result by Jacobi (see e.g. [1, Section 42]) expresses the minors of $\text{Adj}(A)$ in terms of the minors of A .

Lemma 9.2. Let I, J be subsets of $\{1, \dots, n\}$ with $|I| = |J| = r$. We have

$$(9.1) \quad |\text{Adj}(A)|_{I,J} = (-1)^{\sum(I,J)} \det(A)^{r-1} |A|_{J^c, I^c},$$

where $\sum(I, J)$ is the sum of the indices occurring in I and J . □

Definition 9.3. A matrix $A \in \text{GL}_n(\mathbb{C})$ is *generic* if $|A|_{\{k, \dots, n\}, \{1, \dots, n-k+1\}} \neq 0$ for all $k \in \{1, \dots, n\}$.

Note that A is generic if and only if the Ptolemy coordinates of (N, AN) are non-zero. The following is a generalization of Zickert [28, Lemma 3.5].

Proposition 9.4. Let $A \in \text{GL}_n(\mathbb{C})$ be generic. There exist unique $x \in N$ and $y \in N$ such that $q = x^{-1}Ay$ is counter diagonal. The entries of x , y and q are given by

$$(9.2) \quad q_{n,1} = A_{n,1}, \quad q_{n-j+1,j} = (-1)^{j-1} \frac{|A|_{\{n-j+1, \dots, n\}, \{1,j\}}}{|A|_{\{n-j+2, \dots, n\}, \{1, \dots, j-1\}}} \text{ for } 1 < j \leq n$$

$$(9.3) \quad x_{ij} = \frac{|A|_{\{i, j+1, \dots, n\}, \{1, \dots, n-j+1\}}}{|A|_{\{j, \dots, n\}, \{1, \dots, n-j+1\}}} \text{ for } j > i$$

$$(9.4) \quad y_{ij} = (-1)^{i+j} \frac{|A|_{\{n-j+2, \dots, n\}, \{1, \dots, \widehat{i}, \dots, j\}}}{|A|_{\{n-j+2, \dots, n\}, \{1, \dots, j-1\}}} \text{ for } j > i.$$

Proof. It is enough to prove existence and uniqueness of x and y in N such that Ay and $x^{-1}A$ are upper and lower counter triangular, respectively. Suppose Ay is upper counter triangular. Then the vector $y_{\{1, \dots, j-1\}, \{j\}}$ consisting of the part above the counter diagonal of the j th column vector of y must satisfy

$$(9.5) \quad A_{\{n-j+2, \dots, n\}, \{1, \dots, j-1\}} y_{\{1, \dots, j-1\}, \{j\}} + A_{\{n-j+2, \dots, n\}, \{j\}} = 0.$$

The existence and uniqueness of y , as well as the given formula for the entries, now follow from Cramer's rule. Since $x^{-1}A$ is lower counter-triangular if and only if $A^{-1}x$ is upper counter-triangular,

existence and uniqueness of x follows. The explicit formula for the entries follows from Jacobi's identity (9.1) and the formula for the entries of y . To obtain the formula for the entries of q , note that $q_{n-j+1,j} = (Ay)_{n-j+1,j}$. Hence, $q_{n,1} = A_{n,1}$, and for $1 < j \leq n$,

$$\begin{aligned} q_{n-j+1,j} &= \sum_{i=1}^{j-1} A_{n-j+1,i} y_{i,j} + A_{n-j+1,j} \\ &= \frac{\sum_{i=1}^j (-1)^{i+j} A_{n-j+1,i} |A|_{\{n-j+2, \dots, n\}, \{1, \dots, \widehat{i}, \dots, j\}}}{|A|_{\{n-j+2, \dots, n\}, \{1, \dots, j-1\}}} \\ &= (-1)^{j-1} \frac{|A|_{\{n-j+1, \dots, n\}, \{1, \dots, j\}}}{|A|_{\{n-j+2, \dots, n\}, \{1, \dots, j-1\}}}, \end{aligned}$$

where the second equality follows from (9.4). \square

For a generic matrix A , let x_A , y_A and q_A be the unique matrices provided by Proposition 9.4. Given cosets $g_i N$, $g_j N$, $g_k N$, define

$$(9.6) \quad q_{ij} = q_{g_i^{-1} g_j}, \quad \alpha_{jk}^i = (x_{g_i^{-1} g_j})^{-1} x_{g_i^{-1} g_k}.$$

Corollary 9.5. The diagonal left G -action on $C_k^{\mathrm{gen}}(G/N)$ is free when $k \geq 1$, and the chain complex $C_{* \geq 1}^{\mathrm{gen}}(G/N) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ computes relative homology.

Proof. By Proposition 9.4, every generic tuple $(g_0 N, \dots, g_k N)$ may be uniquely written as

$$(9.7) \quad g_0 x_{g_0^{-1} g_1} (N, q_{01} N, \alpha_{12}^0 N, \dots, \alpha_{1k}^0 N).$$

This proves that the G -action is free. Since \mathbb{C} is infinite, there exists for each tuple $(g_0 N, \dots, g_k N)$ a coset gN such that $(gN, g_0 N, \dots, g_k N)$ is generic. Hence, $C_{* \geq 1}^{\mathrm{gen}}(G/N)$ is acyclic, and is thus a free resolution of $\mathrm{Ker}(C_0(G/N) \rightarrow \mathbb{Z})$. This proves the result (see e.g. Zickert [28, Theorem 2.1]). \square

A generic tuple $(g_0 N, \dots, g_k N)$ determines a (G, N) -cocycle on a truncated simplex $\overline{\Delta}$, by labeling the long edges by q_{ij} and the short edges by α_{jk}^i (see Figure 5). In particular, a generically decorated representation ρ determines a (G, N) -cocycle representing ρ .

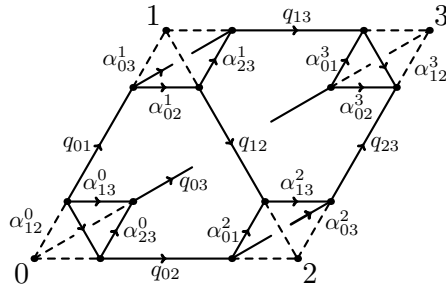


FIGURE 5. A (G, N) -cocycle on a truncated 3-simplex.

A (G, N) -cocycle τ coming from a generic tuple is called *generic*. Letting $\overline{B}_*^{\mathrm{gen}}(G, N)$ be the subcomplex of $\overline{B}_*(G, N)$ generated by generic cocycles on a standard simplex, it follows that we have a canonical isomorphism

$$(9.8) \quad \overline{B}_*^{\mathrm{gen}}(G, N) = C_*^{\mathrm{gen}}(G/N) \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$

It now follows from Proposition 4.7, that the fundamental class can be represented as in (4.3).

We wish to prove that a generic (G, N) -cocycle is uniquely determined by the Ptolemy coordinates.

Notation 9.6. Let $k \in \{1, \dots, n-1\}$.

- (i) For $a_1, \dots, a_n \in \mathbb{C}^*$, let $q(a_1, \dots, a_n)$ be the counter-diagonal matrix whose entries on the counter-diagonal (from lower left to upper right) are a_1, \dots, a_n .
- (ii) For $x \in \mathbb{C}$, let $x_k(x)$ be the elementary matrix whose $(k, k+1)$ entry is x .
- (iii) For $b_1, \dots, b_k \in \mathbb{C}$, let $\pi(b_1, \dots, b_k) = x_1(b_1)x_2(b_2) \cdots x_k(b_k)$.

Lemma 9.7. The long edges of a generic (G, N) -cocycle are determined by the Ptolemy coordinates as follows:

$$(9.9) \quad q_{ij} = q(a_1, \dots, a_n), \quad a_k = (-1)^{k-1} \frac{c_{(n-k)e_i + ke_j}}{c_{(n-k+1)e_i + (k-1)e_j}}.$$

Proof. Let (g_0N, \dots, g_kN) be a generic tuple, and let $A = g_i^{-1}g_j$. Then $q_{ij} = q_A$. Since

$$(9.10) \quad |A|_{\{n-j+1, \dots, n\}, \{1, j\}} = \det(\{g_i\}_{n-k}, \{g_j\}_k) = c_{(n-k)e_i + ke_j},$$

the result follows from (9.2). \square

The corresponding formula for the short edges requires considerable more work, and is given in Lemma 9.12 below.

Lemma 9.8. Let A be generic, and let $L = x_A^{-1}A$. The entries $L_{i, n-i+2}$ right below the counter diagonal are given by

$$(9.11) \quad L_{i, n-i+2} = (-1)^{n-i} \frac{|A|_{\{i, \dots, n\}, \{1, \dots, \widehat{n-i+1}, n-i+2\}}}{|A|_{\{i+1, \dots, n\}, \{1, \dots, n-i\}}}.$$

Proof. We proceed as in the proof of Proposition 9.4. Since L is lower counter-triangular, we must have

$$(9.12) \quad x_{\{i\}, \{i+1, \dots, n\}} A_{\{i+1, \dots, n\}, \{1, \dots, n-i\}} + A_{\{j\}, \{1, \dots, n-i\}} = 0,$$

so by Cramer's rule,

$$(9.13) \quad x_{ij} = (-1)^{i+j} \frac{|A|_{\{j, \dots, \widehat{j}, \dots, n\}, \{1, \dots, n-i\}}}{|A|_{\{i+1, \dots, n\}, \{1, \dots, n-i\}}} \text{ for } j > i.$$

We thus have

$$\begin{aligned} |A|_{\{i+1, \dots, n\}, \{1, \dots, n-i\}} L_{i, n-i+2} &= A_{i, n-i+2} |A|_{\{i+1, \dots, n\}, \{1, \dots, n-i\}} \\ &\quad + \sum_{k=i+1}^n (-1)^{i+k} |A|_{\{j, \dots, \widehat{k}, \dots, n\}, \{1, \dots, n-j\}} A_{k, n-i+2} \\ &= \sum_{k=j}^n (-1)^{i+k} |A|_{\{j, \dots, \widehat{k}, \dots, n\}, \{1, \dots, n-i\}} A_{k, n-i+2} \\ &= (-1)^{n-i} |A|_{\{i, \dots, n\}, \{1, \dots, \widehat{n-i+1}, \dots, n-i+2\}}, \end{aligned}$$

which proves the result. \square

Definition 9.9. Let $A, B \in \text{GL}(n, \mathbb{C})$.

- (i) A and B are related by a *type 0 move* if all but the last column of A and B are equal.

- (ii) A and B are related by a *type 1 move* if all but the second last column of A and B are equal.
- (iii) A and B are related by a *type 2 move* if for some $j < n-1$, B is obtained from A by switching columns j and $j+1$.

Proposition 9.10. Let A and B be generic, and let A_i and B_i denote the i th column of A_i , resp. B_i .

- (i) If A and B are related by a type 0 move, $x_B = x_A$.
- (ii) If A and B are related by a type 1 move, $x_B = x_A x_1(x)$, where

$$(9.14) \quad x = -\frac{\det(A_1, \dots, A_{n-1}, B_{n-1}) \det(e_1, e_2, A_1, \dots, A_{n-2})}{\det(e_1, A_1, \dots, A_{n-1}) \det(e_1, A_1, \dots, A_{n-2}, B_{n-1})}.$$

- (iii) If A and B are related by a type 2 move switching columns j and $j+1$, we have $x_B = x_A x_{n-j}(x)$, where

$$(9.15) \quad x = -\frac{\det(e_1, \dots, e_{n-j-1}, A_1, \dots, A_{j+1}) \det(e_1, \dots, e_{n-j+1}, A_1, \dots, A_{j-1})}{\det(e_1, \dots, e_{n-j}, A_1, \dots, A_j) \det(e_1, \dots, e_{n-j}, A_1, \dots, A_{j-1}, B_j)}.$$

Proof. The first statement follows from the fact that x_A is independent of the last column of A . Suppose A and B are related by a type 1 move. Using (9.3), one sees that $(x_A)_{ij} = (x_B)_{ij}$ except when $\{i, j\} = \{1, 2\}$. It thus follows that $x_B = x_A x_1(x)$, where $x = (x_B)_{12} - (x_A)_{12}$. Letting C be the matrix obtained from A by replacing the n th column by the $(n-1)$ th column of B , one has

$$\begin{aligned} |A|_{\{1,3,\dots,n\},\{1,\dots,n-1\}} &= \mathrm{Adj}(C)_{n,2}, & |B|_{\{1,3,\dots,n\},\{1,\dots,n-1\}} &= \mathrm{Adj}(C)_{n-1,2}, \\ |A|_{\{2,\dots,n\},\{1,\dots,n-1\}} &= \mathrm{Adj}(C)_{n,1}, & |B|_{\{2,\dots,n\},\{1,\dots,n-1\}} &= \mathrm{Adj}(C)_{n-1,1}, \end{aligned}$$

and it follows from (9.3) that

$$(9.16) \quad x = (x_B)_{12} - (x_A)_{12} = \frac{\mathrm{Adj}(C)_{n-1,2}}{\mathrm{Adj}(C)_{n-1,1}} - \frac{\mathrm{Adj}(C)_{n,2}}{\mathrm{Adj}(C)_{n,1}}.$$

We then have

$$\begin{aligned} x \mathrm{Adj}(C)_{n,1} \mathrm{Adj}(C)_{n-1,1} &= \mathrm{Adj}(C)_{n-1,2} \mathrm{Adj}(C)_{n,1} - \mathrm{Adj}(C)_{n-1,1} \mathrm{Adj}(C)_{n,2} \\ &= -|\mathrm{Adj}(C)|_{\{n-1,n\},\{1,2\}} \\ &= -\det(C) |C|_{\{3,\dots,n\},\{1,\dots,n-2\}} \\ &= -\det(A_1, \dots, A_{n-1}, B_{n-1}) \det(e_1, e_2, A_1, \dots, A_{n-2}), \end{aligned}$$

where the third equality follows from Jacobi's identity (9.1). Since

$$\mathrm{Adj}(C)_{n,1} \mathrm{Adj}(C)_{n-1,1} = \det(e_1, A_1, \dots, A_{n-1}) \det(e_1, A_1, \dots, A_{n-2}, B_{n-1}),$$

this proves the second statement.

Now suppose A and B are related by a type 2 move. Let $E_{j,j+1}$ be the elementary matrix obtained from the identity matrix by switching the j th and $(j+1)$ th columns. Then $B = AE_{j,j+1}$. Since $L = x_A^{-1}A$ is lower counter triangular, $x_{n-j}(-\frac{L_{n-j,j+1}}{L_{n-j+1,j+1}})LE_{j,j+1}$ must also be lower counter triangular. We thus have

$$(9.17) \quad x_B = x_A x_{n-j}\left(-\frac{L_{n-j,j+1}}{L_{n-j+1,j+1}}\right) = x_A x_{n-j}\left(\frac{L_{n-j,j+1}}{L_{n-j+1,j+1}}\right).$$

By (9.11) and (9.2), we have

$$(9.18) \quad \begin{aligned} L_{n-j+1,j+1} &= (-1)^{j-1} \frac{|A|_{\{n-j+1,\dots,n\},\{1,\dots,\widehat{j},j+1\}}}{|A|_{\{n-j+2,\dots,n\},\{1,\dots,j-1\}}} \\ L_{n-j,j+1} &= (-1)^j \frac{|A|_{\{n-j,\dots,n\},\{1,\dots,j+1\}}}{|A|_{\{n-j+1,\dots,n\},\{1,\dots,j\}}}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{L_{n-j,j+1}}{L_{n-j+1,j+1}} &= - \frac{|A|_{\{n-j,\dots,n\},\{1,\dots,j+1\}} |A|_{\{n-j+2,\dots,n\},\{1,\dots,j-1\}}}{|A|_{\{n-j+1,\dots,n\},\{1,\dots,j\}} |A|_{\{n-j+1,\dots,n\},\{1,\dots,\widehat{j},j+1\}}} \\ &= - \frac{\det(e_1, \dots, e_{n-j-1}, A_1, \dots, A_{j+1}) \det(e_1, \dots, e_{n-j+1}, A_1, \dots, A_{j-1})}{\det(e_1, \dots, e_{n-j}, A_1, \dots, A_j) \det(e_1, \dots, e_{n-j}, A_1, \dots, A_{j-1}, B_j)}, \end{aligned}$$

proving the third statement. \square

Note that any two matrices $A, B \in \text{GL}(n, \mathbb{C})$ are related by a sequence of moves of type 1, 2 and 0 as follows:

$$(9.19) \quad \begin{aligned} A &\xrightarrow{1} [A_1, \dots, A_{n-2}, B_1, A_n] \xrightarrow{2} [A_1, \dots, A_{n-3}, B_1, A_{n-2}, A_n] \xrightarrow{2} \dots \xrightarrow{2} \\ &[B_1, A_1, \dots, A_{n-2}, A_n] \xrightarrow{1} [B_1, A_1, \dots, A_{n-3}, B_2, A_n] \xrightarrow{2} \dots \xrightarrow{2} \\ &[B_1, B_2, A_1, \dots, A_{n-3}, A_n] \xrightarrow{1,2} \dots \xrightarrow{1,2} [B_1, \dots, B_{n-1}, A_n] \xrightarrow{0} B. \end{aligned}$$

Consider the tilings of a face ijk , $i < j < k$, of Δ_n^2 by *diamonds* shown in Figure 6. We refer to the diamonds as being of type i , j and k , respectively.

Definition 9.11. The *diamond coordinate* $d_{r,s}^k$ of a diamond (r,s) of type k is the alternating product of the Ptolemy coordinates assigned to its vertices, see Figure 6.

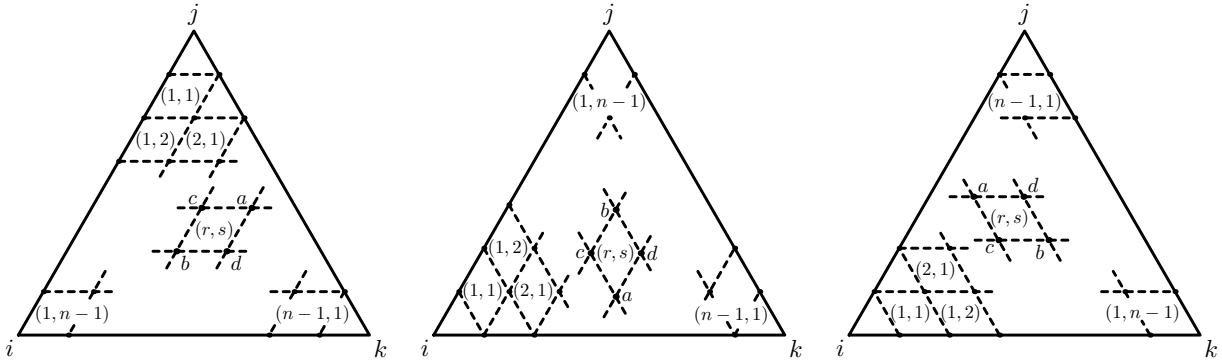


FIGURE 6. Diamonds of type i , j and k . The diamond coordinates are $d_{r,s}^i = d_{r,s}^k = \frac{-ab}{cd}$, and $d_{r,s}^j = \frac{ab}{cd}$, where a , b , c , and d are Ptolemy coordinates.

Lemma 9.12. The short edges α_{jk}^i , $j < k$, of a generic (G, N) -cocycle are determined by the Ptolemy coordinates as follows:

$$(9.20) \quad \alpha_{jk}^i = \pi_{n-1}(d_{1,1}^i, \dots, d_{1,n-1}^i) \pi_{n-2}(d_{2,1}^i, \dots, d_{2,n-2}^i) \cdots \pi_1(d_{n-1,1}^i),$$

where the $d_{j,k}^i$'s are the type i diamond coordinates on the face ijk .

Proof. Let (g_0N, \dots, g_lN) be a generic tuple, and let $A = g_i^{-1}g_j$ and $B = g_i^{-1}g_k$. We assume that $i < j < k$, the other cases being similar. Note that the Ptolemy coordinates on the ijk face are given by

$$(9.21) \quad c_{t_i e_i + t_j e_k + t_k e_k} = \det(e_1, \dots, e_{t_i}, A_1, \dots, A_{t_j}, B_1, \dots, B_{t_k}).$$

Performing the sequence of moves in (9.19), the result follows from Proposition 9.10. \square

Corollary 9.13. A generic tuple is determined up to the diagonal G -action by its Ptolemy coordinates. \square

This result together with Lemma 9.7 shows that there is at most one generic (G, N) -cocycle with a given collection of Ptolemy coordinates. We now prove that the Ptolemy relations are the only relations among the Ptolemy coordinates when $k \leq 3$.

Example 9.14. Suppose Ptolemy cochains on Δ_n^2 , $n \in \{2, 3\}$, are given as in Figure 7. Using (9.9) and (9.20), we obtain that the corresponding (G, N) -cocycle is given by

$$(9.22) \quad \begin{aligned} q_{01} &= q(a, -1/a), & q_{12} &= q(b, -1/b), & q_{02} &= q(c, -1/c), \\ \alpha_{12}^0 &= x_1\left(\frac{-b}{ac}\right), & \alpha_{02}^1 &= x_1\left(\frac{c}{ab}\right), & \alpha_{01}^2 &= x_1\left(\frac{-a}{cb}\right) \end{aligned}$$

when $n = 2$, and

$$(9.23) \quad \begin{aligned} q_{01} &= q(c, -a/c, 1/a), & q_{12} &= q(b, -e/b, 1/e), & q_{02} &= q(f, -g/f, 1/g), \\ \alpha_{02}^1 &= x_1\left(\frac{fa}{cd}\right)x_2\left(\frac{d}{ab}\right)x_1\left(\frac{gb}{de}\right), \\ \alpha_{12}^0 &= x_1\left(\frac{-bc}{ad}\right)x_2\left(\frac{-d}{cf}\right)x_1\left(\frac{-ef}{dg}\right), & \alpha_{01}^2 &= x_1\left(\frac{-cg}{fd}\right)x_2\left(\frac{-d}{ge}\right)x_1\left(\frac{-ae}{db}\right) \end{aligned}$$

when $n = 3$.

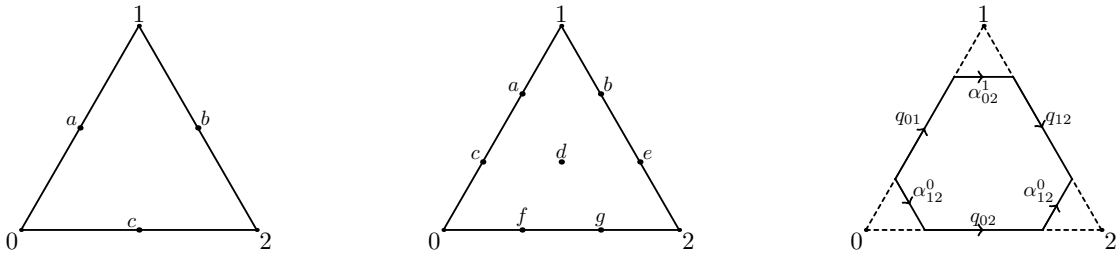


FIGURE 7. Ptolemy cochains and the corresponding cocycle for $n = 2$ and $n = 3$.

Lemma 9.15. Let $a_{i,j}$ and $b_{i,j}$ be non-zero complex numbers. The equality

$$(9.24) \quad \pi_{n-1}(a_{1,1}, \dots, a_{1,n-1}) \cdots \pi_1(a_{n-1,1}) = \pi_{n-1}(b_{1,1}, \dots, b_{1,n-1}) \cdots \pi_1(b_{n-1,1})$$

holds if and only if $a_{i,j} = b_{i,j}$ for all i, j .

Proof. For any $c_{i,j}$, the n th column of $\pi_{n-1}(c_{1,1}, \dots, c_{1,n-1}) \cdots \pi_1(c_{n-1,1})$ is equal to the n th column of $\pi_{n-1}(c_{1,1}, \dots, c_{1,n-1})$, which equals

$$\left(\prod_{i=1}^{n-1} c_{1,i}, \prod_{i=2}^{n-1} c_{1,i}, \dots, c_{1,n-1} \right).$$

This proves that $a_{1,j} = b_{1,j}$ for all j . The result now follows by induction. \square

Proposition 9.16. For any assignment $c: \mathring{T}^2(n) \rightarrow \mathbb{C}^*$, there is a unique Ptolemy cochain $c \in Pt_2^n$ whose Ptolemy coordinates are c_t .

Proof. We prove that the Ptolemy coordinates c'_t of $(N, q_{01}N, \alpha_{12}^0 q_{02}N)$ equal c_t , where q_{01}, q_{02} and α_{12}^0 are given in terms of the c_t 's by (9.9) and (9.20). First note that $c_t = c'_t$ if either t_1 or t_2 is 0, i.e. if t is on one of the edges of Δ_n^2 containing the 0th vertex. Each of the other integral points t is the upper right vertex of a unique diamond (r, s) of type 0. Let τ_k be the upper right vertex of the k th diamond D_k in the sequence

$$(9.25) \quad (1, n-1), (1, n-2), \dots, (1, 1), (2, n-2), \dots, (2, 1), \dots, (n-1, 1).$$

By Lemma 9.15, $d_{r,s}^{0'} = d_{r,s}^0$ for all diamonds (r, s) of type 0. It thus follows that if $c_t = c'_t$ for all but one of the vertices of a diamond D , then $c_t = c'_t$ for all vertices of D . In particular $c'_{\tau_1} = c_{\tau_1}$. Suppose by induction that $c'_{\tau_i} = c_{\tau_i}$ for all $i < k$. Then $c'_t = c_t$, for all vertices of D_k except τ_k . Hence, we also have $c'_{\tau_k} = c_{\tau_k}$, completing the induction. \square

Proposition 9.17. For any assignment $c: \mathring{T}^3(n) \rightarrow \mathbb{C}^*$ satisfying the Ptolemy relations, there is a unique Ptolemy cochain $c \in Pt_3^n$ whose Ptolemy coordinates are c_t .

Proof. Let c'_t be the Ptolemy coordinates of the tuple $(N, q_{01}N, \alpha_{12}^0 q_{02}N, \alpha_{13}^0 q_{03}N)$ defined from the c_t 's by (9.9) and (9.20). We wish to prove that $c'_t = c_t$ for all t . Note that if, for some subsimplex $\Delta^3(\alpha)$, $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ for all but one of the 6 α_{ij} 's, then $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ holds for all α_{ij} . This is a direct consequence of the Ptolemy relations. By Proposition 9.16, $c'_t = c_t$, when either t_2 or t_3 is zero. Hence, for each $\alpha = (a_0, a_1, a_2, a_3)$ with $a_2 = a_3 = 0$, $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ except possibly when $(i, j) = (2, 3)$. As explained above, $c'_{\alpha_{23}} = c_{\alpha_{23}}$ as well. Now suppose by induction that $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ for all α with $a_2 + a_3 < k$. Then $c'_{\alpha_{ij}} = c_{\alpha_{ij}}$ holds except possibly when $(i, j) = (2, 3)$. Again, $c'_{\alpha_{23}} = c_{\alpha_{23}}$ must also hold, completing the induction. \square

A (G, N) -cocycle on M obviously determines a decorated representation (up to conjugation). The main results of this section can thus be summarized by the diagram below.

$$(9.26) \quad \left\{ \text{Points in } P_n(K) \right\} \longleftrightarrow \left\{ \text{Generic } (G, N)\text{-cocycles} \right. \\ \left. \text{on } M \right\} \longleftrightarrow \left\{ \text{Generically decorated} \right. \\ \left. (G, N)\text{-representations} \right\}$$

Remark 9.18. We stress that the Ptolemy variety parametrizes decorated representations on K and *not* decorated representations up to equivalence. In particular, the dimension of $P(K)$ depends on the triangulation, and may be very large if K has many interior vertices.

Remark 9.19. For $n = 3$, coordinates parametrizing 3-cycles decorated by flags have also been considered (independently) by Bergeron, Falbel and Guilloux [2]. An alternative parametrization of representations is given by Kashaev's Δ -groupoid [17].

9.1. Obstruction cocycles and the $p\text{SL}(n, \mathbb{C})$ -Ptolemy varieties. Suppose n is even. The projection $G \rightarrow pG$ maps N isomorphically onto its image (also denoted by N), and by elementary obstruction theory (see e.g. Steenrod [22]), the obstruction to lifting a (pG, N) -representation ρ to a (G, N) -representation is a class in

$$(9.27) \quad H^2(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = H^2(K; \mathbb{Z}/2\mathbb{Z}).$$

We can represent it by an explicit cocycle in $Z^2(K; \mathbb{Z}/2\mathbb{Z})$ as follows: Pick any $(p\text{SL}(n, \mathbb{C}), N)$ -cocycle $\bar{\tau}$ on M representing ρ and a lift τ of $\bar{\tau}$ to a (G, N) -cochain. Each 2-cell of K corresponds

to a hexagonal 2-cell of M , and the 2-cocycle $\sigma \in Z^2(K; \mathbb{Z}/2\mathbb{Z})$ taking a 2-cell to the product of the τ -labelings along the corresponding hexagonal 2-cell of M represents the obstruction class.

Proposition 9.20. Suppose the interior of M is a cusped hyperbolic 3-manifold with finite volume. The obstruction class in $H^2(K; \mathbb{Z}/2\mathbb{Z})$ to lifting the geometric representation is non-trivial.

Proof. By a result of Calegari [5, Corollary 2.4], any lift of the geometric representation takes any curve bounding a two-sided incompressible surface to an element in $\mathrm{SL}(2, \mathbb{C})$ with trace -2 . This shows that no lift is boundary-unipotent, so the obstruction class must be non-trivial. \square

Proposition 9.4 also holds in $p\mathrm{SL}(n, \mathbb{C})$, and we thus have a 1-1 correspondence between generically decorated representations and (pG, N) -cocycles on M .

Definition 9.21. Let $\sigma \in Z^2(K; \mathbb{Z}/2\mathbb{Z})$. A lifted (pG, N) cocycle on M with obstruction cocycle σ is a generic (G, N) -cochain on M lifting a (pG, N) -cocycle on M such that the 2-cocycle on K obtained by taking products along hexagonal faces of M equals σ .

A 1-cochain $\eta \in C^1(K; \mathbb{Z}/2\mathbb{Z})$ acts on a lifted (pG, N) -cocycle τ by multiplying a long edge e by $\eta(e)$. Note that if τ has obstruction cocycle σ , $\eta\tau$ has obstruction cocycle $\delta(\eta)\sigma$, where δ is the standard coboundary operator. Recall that there is a 1-1 correspondence between generic (G, N) -cocycles on M and points in the Ptolemy-variety. We shall prove a similar result for pG .

We wish to define a coboundary action on pG -Ptolemy cochains (see Definition 5.11). Let c be a $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy cochain on Δ , and let $\eta_{ij} \in C^1(\Delta; \mathbb{Z}/2\mathbb{Z})$ be the cochain taking the edge ij to -1 and all other edges to 1. Define

$$(9.28) \quad \eta_{ij}c: \mathring{T}^k(n) \rightarrow \mathbb{C}^*, \quad (\eta_{ij}c)_t = \begin{cases} -c_t & \text{if } (-1)^{t_i t_j} = -1 \\ c_t & \text{otherwise,} \end{cases}$$

and extend in the natural way to define ηc for a pG -Ptolemy cochain c on K and $\eta \in C^1(K; \mathbb{Z}/2\mathbb{Z})$. A priori ηc is only an assignment of complex numbers to the integral points of the simplices of K . However, we have:

Lemma 9.22. If c is a pG -Ptolemy cochain on K with obstruction cocycle σ , ηc is a pG -Ptolemy cochain on K with obstruction cocycle $\delta(\eta)\sigma$.

Proof. It is enough to prove this for a simplex Δ and for $\eta = \eta_{ij}$. Let $c' = \eta_{ij}c$. We assume for simplicity that $ij = 01$; the other cases are proved similarly. For any $\alpha = (a_0, a_1, a_2, a_3) \in T^k(n)$, we then have

$$(9.29) \quad c'_{\alpha_{03}} c'_{\alpha_{12}} + c'_{\alpha_{01}} c'_{\alpha_{23}} - c'_{\alpha_{02}} c'_{\alpha_{13}} = \begin{cases} +c_{\alpha_{03}} c_{\alpha_{12}} - c_{\alpha_{01}} c_{\alpha_{23}} - c_{\alpha_{02}} c_{\alpha_{13}} & \text{if } (-1)^{a_0 + a_1} = 1 \\ -c_{\alpha_{03}} c_{\alpha_{12}} + c_{\alpha_{01}} c_{\alpha_{23}} + c_{\alpha_{02}} c_{\alpha_{13}} & \text{if } (-1)^{a_0 + a_1} = -1. \end{cases}$$

Let $\tau = \delta(\eta_{01})$. Since $\delta(\eta_{01})_2 = \delta(\eta_{01})_3 = -1$ and $\delta(\eta_{01})_0 = 1$, (9.29) implies that

$$(9.30) \quad \tau_2 \tau_3 c'_{\alpha_{03}} c'_{\alpha_{12}} + \tau_0 \tau_3 c'_{\alpha_{03}} c'_{\alpha_{01}} c'_{\alpha_{23}} = c'_{\alpha_{02}} c'_{\alpha_{13}},$$

as desired. \square

Definition 9.23. The diamond coordinates of a $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy cochain with obstruction cocycle σ are defined as in Definition 9.11, but multiplied by the sign (provided by σ) of the face.

Note that for $\eta \in C^1(K; \mathbb{Z}/2\mathbb{Z})$, the diamond coordinates of c and ηc are identical.

Proposition 9.24. For any $\sigma \in Z^2(K; \mathbb{Z}/2\mathbb{Z})$, there is a 1-1 correspondence between $p\mathrm{SL}(n, \mathbb{C})$ -Ptolemy cochains on K with obstruction cocycle σ , and lifted $(p\mathrm{SL}(n, \mathbb{C}), N)$ -cocycles on M with obstruction cocycle σ . The correspondence preserves the coboundary actions.

Proof. It is enough to prove this for a simplex Δ . For a pG -Ptolemy cochain c on Δ with obstruction cocycle $\sigma \in Z^2(\Delta; \mathbb{Z}/2\mathbb{Z})$, define a cochain τ on $\overline{\Delta}$ by the formulas (9.9) and (9.20) using the σ -modified diamond coordinates (Definition 9.23). Let $\eta \in C^1(\Delta; \mathbb{Z}/2\mathbb{Z})$ be such that $\delta\eta = \sigma$, where δ is the standard coboundary map. By Lemma 9.22 ηc satisfies the $\mathrm{SL}(n, \mathbb{C})$ Ptolemy relations (5.5), and hence corresponds to an $(\mathrm{SL}(n, \mathbb{C}), N)$ -cocycle τ' . Since the diamond coordinates of c and ηc are the same, the short edges of τ' agree with those of τ and the long edges differ from those of τ by δ . This proves that τ is a lifted (pG, N) -cocycle with obstruction cocycle σ . The inductive arguments of Propositions 9.16 and 9.17 show that this is a 1-1 correspondence. The fact that actions by coboundaries correspond is immediate from the construction. \square

Corollary 9.25. Let $\sigma \in Z^2(K; \mathbb{Z}/2\mathbb{Z})$. There is an algebraic variety $P_n^\sigma(K)$ of generically decorated boundary-unipotent representations $\rho: \pi_1(M) \rightarrow p\mathrm{SL}(n, \mathbb{C})$ whose obstruction class to lifting to $\mathrm{SL}(n, \mathbb{C})$ is represented by σ . Up to canonical isomorphism, the variety $P_n^\sigma(K)$ only depends on the cohomology class of σ .

Proof. This follows immediately from Proposition 9.24. \square

Note that the canonical isomorphisms in Corollary 9.25 respect the extended Bloch group element. This follows from the pG variant of Proposition 7.7. As in (9.26) we have

$$\left\{ \text{Points in } P_n^\sigma(K) \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Lifted } (pG, N)\text{-cocycles on } M \\ \text{with obstruction cocycle } \sigma \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Generically decorated} \\ (pG, N)\text{-representations} \\ \text{with obstruction class } \sigma \end{array} \right\}$$

9.2. Proof of Theorem 1.3 and Theorem 1.8. Let $\rho: P_n(K) \rightarrow R_{G,N}(M)/\mathrm{Conj}$ be the composition of the map in (9.26) with the forgetful map ignoring the decoration. Let $c \in P_n(K)$. By Proposition 6.12, $\lambda(c)$ is in $\widehat{\mathcal{B}}(\mathbb{C})$ and by Proposition 7.8, $\lambda(c)$ only depends on $\rho(c)$. Commutativity of diagram (1.10) follows from Remark 8.4, and the fact that ρ is surjective if K is sufficiently fine follows from Proposition 5.4. This concludes the proof of Theorem 1.3. The first part of Theorem 1.8 is proved similarly, and the last part follows from Theorem 11.7 below.

10. EXAMPLES

In the examples below, all computations of Ptolemy varieties are exact, whereas the computations of complex volume are numerical with at least 50 digits precision.

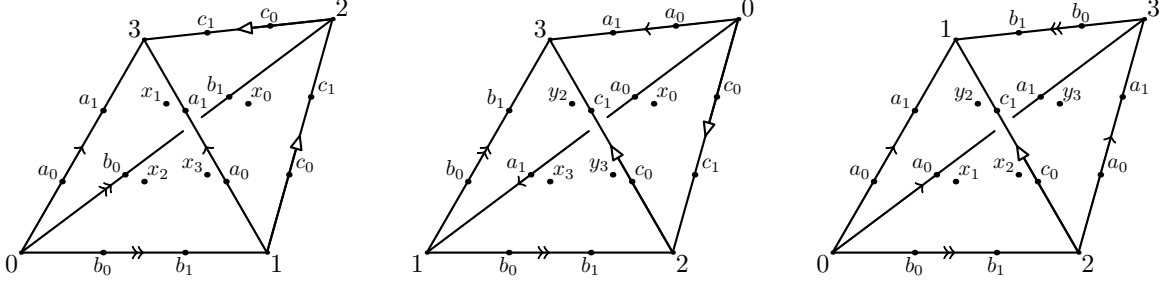
Example 10.1 (The 5_2 knot complement). Consider the 3-cycle K obtained from the simplices in Figure 8 by identifying the faces via the unique simplicial attaching maps preserving the arrows. The space obtained from K by removing the 0-cell is homeomorphic to the complement of the 5_2 knot, as is verified by [9].

Labeling the Ptolemy coordinates as in Figure 8, the Ptolemy variety for $n = 3$ is given by the equations

$$(10.1) \quad \begin{array}{lll} a_0x_3 + b_0x_1 = b_0x_2, & a_0y_3 + a_0x_0 = c_0y_2, & a_0x_2 + b_0y_2 = a_0x_1 \\ x_2c_0 + b_1x_0 = x_3a_0, & y_2b_0 + a_1x_3 = y_3b_0, & x_1a_0 + b_1y_3 = x_2c_0 \\ x_1c_1 + x_3c_0 = b_1x_0, & x_0b_1 + y_3c_0 = c_1x_3, & y_2a_1 + x_2b_0 = a_1y_3 \\ a_1x_0 + x_2c_1 = x_1a_1, & a_1x_3 + y_2c_1 = x_0b_1, & a_1y_3 + x_1b_1 = y_2c_1 \end{array}$$

together with an extra equation (involving an additional variable t)

$$(10.2) \quad a_0a_1b_0b_1c_0c_1x_0x_1x_2x_3y_2y_3t = 1,$$

FIGURE 8. A 3-cycle structure on the 5_2 knot complement, and Ptolemy coordinates for $n = 3$.

making sure that all Ptolemy coordinates are non-zero. By Remark 4.5, the diagonal matrices act on the decorations, and one easily checks that the action by a matrix $\text{diag}(x, y, z)$ with determinant 1 multiplies a Ptolemy coordinate on an edge by x^2y and a Ptolemy coordinate on a face by x^3 . Since we are not interested in the particular decoration, we may thus assume e.g. that $a_0 = y_3 = 1$. Using Magma [3], one finds that the Ptolemy variety, after setting $a_0 = y_3 = 1$, has three 0-dimensional components with 3, 4 and 6 points respectively. One of these is given by

$$(10.3) \quad \begin{aligned} a_0 = a_1 = y_3 = 1, \quad x_1 = -1, \quad c_0 = c_1 = x_0^2 + 2x_0 + 1 \\ y_2 = x_0^2 + 2 = -x_2, \quad x_3 = -x_0^2 - x_0 - 1 \\ x_0^3 + x_0^2 + 2x_0 + 1 = 0 \end{aligned}$$

Thus, this component gives rise to 3 representations, one for each solution to $x_0^3 + x_0^2 + 2x_0 + 1 = 0$. Using the fact that $R(\lambda(c)) = i \text{Vol}_{\mathbb{C}}(\rho)$, the complex volumes of these can be computed to be

$$(10.4) \quad 0.0 - 4.453818209 \dots i \in \mathbb{C}/4\pi^2 i\mathbb{Z}, \quad \pm 11.31248835 \dots + 12.09651350 \dots i \in \mathbb{C}/4\pi^2 i\mathbb{Z}$$

corresponding to the values $x_0 = -0.5698 \dots$ and $x_0 = -0.2150 \mp 1.3071 \dots i$, respectively.

In Zickert [28, Section 6], the complex volumes of the Galois conjugates of the geometric representation are computed to be

$$(10.5) \quad 0.0 - 1.113454552 \dots i \in \mathbb{C}/\pi^2 i\mathbb{Z}, \quad \pm 2.828122088 \dots + 3.024128376 \dots i \in \mathbb{C}/\pi^2 i\mathbb{Z}.$$

Notice that (10.4) is (approximately) 4 times (10.5). It thus follows from Theorem 1.6 that the representations given by (10.3) are ϕ_3 composed with the geometric component of $PSL(2, \mathbb{C})$ -representations and that the factor of 4 is exact.

Another component is given by

$$(10.6) \quad \begin{aligned} a_0 = a_1 = y_3 = 1, \quad x_1 = -1, \quad b_1 = -x_0 \\ b_0 = 1/4x_0^3 - 1/4x_0^2 + 3/4x_0 - 1/2 \\ c_0 = c_1 = 1/4x_0^3 - 1/4x_0^2 - 1/4x_0 + 1/2 \\ y_2 = -x_2 = 1/4x_0^3 + 3/4x_0^2 + 7/4x_0 + 3/2 \\ x_3 = -x_0^2 - x_0 - 1 \\ x_0^4 + x_0^3 + x_0^2 - 4x_0 - 4 = 0. \end{aligned}$$

In this case there are two distinct complex volumes given by:

$$(10.7) \quad 0.0 + 2.631894506 \dots i = \frac{4}{15}\pi^2 i \in \mathbb{C}/4\pi^2 i\mathbb{Z}, \quad 0.0 + 10.527578027 \dots i = \frac{16}{15}\pi^2 i \in \mathbb{C}/4\pi^2 i\mathbb{Z}.$$

The third component has somewhat larger coefficients, but after introducing a variable u with $u^6 + 5u^4 + 8u^2 - 2u + 1 = 0$, the defining equations simplify to

$$\begin{aligned}
 (10.8) \quad & a_0 = y_3 = 1, \quad a_1 = 1/4u^5 + 1/4u^4 + 5/4u^3 + 1/2u^2 + 2u - 3/4 \\
 & b_0 = b_1 = -1/4u^4 - 3/4u^2 - 1/4u - 3/4, \\
 & c_1 = -1/4u^5 - 3/4u^3 - 1/4u^2 - 3/4u, \\
 & c_0 = 1/2u^5 + 9/4u^3 + 1/4u^2 + 7/2u - 1/4, \\
 & y_2 = -8/17u^5 - 1/34u^4 - 79/34u^3 - 3/17u^2 - 105/34u + 26/17, \\
 & x_3 = 1/17u^5 - 1/17u^4 + 6/17u^3 - 6/17u^2 + 14/17u - 16/17, \\
 & x_2 = 9/34u^5 + 4/17u^4 + 37/34u^3 + 31/34u^2 + 75/34u + 13/17, \\
 & x_1 = 8/17u^5 + 1/34u^4 + 79/34u^3 + 3/17u^2 + 139/34u - 9/17, \\
 & x_0 = 15/34u^5 + 1/17u^4 + 73/34u^3 + 29/34u^2 + 125/34u - 1/17, \\
 & u^6 + 5u^4 + 8u^2 - 2u + 1 = 0.
 \end{aligned}$$

In this case, there are 3 distinct complex volumes:

$$(10.9) \quad 0.0 + 1.241598704 \dots i, \quad \pm 6.332666642 \dots + 1.024134714 \dots i$$

According to Conjecture 1.15, $6.33 \dots + 1.02 \dots i$ should (up to rational multiples of $\pi^2 i$) be an integral linear combination of complex volumes of hyperbolic manifolds. Using e.g. Snap [15], one checks that the complex volume of the manifold $m034$ is given by

$$(10.10) \quad 3.166333321 \dots + 2.157001424 \dots i,$$

and we have

$$(10.11) \quad 6.332666642 \dots + 1.024134714 \dots i = 2 \operatorname{Vol}_{\mathbb{C}}(m034) - \frac{1}{3} \pi^2 i \in \mathbb{C}/4\pi^2 i \mathbb{Z}.$$

Example 10.2 (The figure 8 knot complement). Let K be the 3-cycle in Figure 9. Then $M = M(K)$ is the figure 8 knot complement, and $H^2(K; \mathbb{Z}/2\mathbb{Z}) = H^2(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

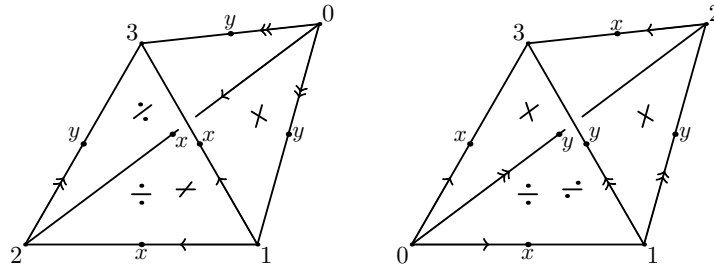


FIGURE 9. A 3-cycle structure on the figure 8 knot complement and Ptolemy coordinates for $n = 2$. The signs indicate the non-trivial second $\mathbb{Z}/2\mathbb{Z}$ cohomology class.

For the trivial obstruction class, the Ptolemy variety for $n = 2$ is given by

$$(10.12) \quad yx + y^2 = x^2, \quad xy + x^2 = y^2,$$

and is thus empty since x and y are non-zero. In fact, the only boundary-unipotent representation in $\operatorname{SL}(2, \mathbb{C})$ is the trivial representation, so this is not surprising. The non-trivial obstruction class

can be represented by the cocycle indicated in Figure 9, and the Ptolemy variety is given by

$$(10.13) \quad yx - y^2 = x^2, \quad xy - x^2 = y^2.$$

As in Example 10.1, we may assume $y = 1$. Hence, the Ptolemy variety detects two (complex conjugate) representations corresponding to the solutions to $x^2 - x + 1 = 0$. The extended Bloch group elements are

$$(10.14) \quad -(-\tilde{x}, -2\tilde{x}) + (\tilde{x}, 2\tilde{x}) \in \widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}},$$

with complex volume

$$(10.15) \quad \pm 2.029883212 \dots + 0.0i.$$

We thus recover the well known complex volume of the figure 8 knot complement.

For $n = 3$, similar calculations as those in Example 10.1 show that the only representations in $\mathrm{SL}(3, \mathbb{C})$ detected by the Ptolemy variety, are those induced by the geometric representation. For $n = 4$, lots of new complex volumes emerge. For the trivial obstruction class, the non-zero complex volumes are

$$(10.16) \quad \pm 7.327724753 \dots + 0.0i = 2 \mathrm{Vol}_{\mathbb{C}}(5_1^2) + \pi^2 i/4,$$

where the manifold 5_1^2 is the whitehead link complement. For the non-trivial obstruction class, the complex volumes are

$$(10.17) \quad \begin{aligned} &\pm 20.29883212 \dots + 0.0i = 10 \mathrm{Vol}_{\mathbb{C}}(4_1) \in \mathbb{C}/\pi^2 i\mathbb{Z} \\ &\pm 4.260549384 \dots \pm 0.136128165 \dots i \\ &\pm 3.230859569 \dots + 0.0i \\ &\pm 8.355502146 \dots + 2.428571615 \dots i = \mathrm{Vol}_{\mathbb{C}}(-9_{15}^3) + 2\pi^2 i/3 \\ &\pm 3.276320849 \dots + 9.908433886 \dots i. \end{aligned}$$

Example 10.3 ($S^1 \times S^2$). Figure 10 shows a triangulation of $M = S^1 \times S^2$ taken from the Regina census [4]. Since $\pi_1(S^1 \times S^2) = \mathbb{Z}$, all representations in $\mathrm{PSL}(2, \mathbb{C})$ lift to $\mathrm{SL}(2, \mathbb{C})$, so we expect the Ptolemy variety for the non-trivial class in $H^2(M; \mathbb{Z}/2\mathbb{Z})$ to be zero. This class is represented by the cocycle shown in Figure 10, and the Ptolemy variety is given by

$$(10.18) \quad -zx + x^2 = y^2, \quad x^2 + zx = y^2,$$

which indeed has no solutions in \mathbb{C}^* . For the trivial cohomology class, all signs are positive, and the two equations are equivalent. The extended Bloch group element is

$$(10.19) \quad (\tilde{z} + \tilde{x} - 2\tilde{y}, 2\tilde{x} - 2\tilde{y}) - (\tilde{z} + \tilde{x} - 2\tilde{y}, 2\tilde{x} - 2\tilde{y}) = 0 \in \widehat{\mathcal{B}}(\mathbb{C}).$$

In fact, the extended Bloch group element of a Ptolemy cochain is trivial for all n , as one easily verifies (the subsimplices cancel out in pairs).

We wish to find out which representations are detected by $P_2(K)$. A choice of fundamental domain F for K in L determines a presentation of $\pi_1(M)$ with a generator for each face pairing of F and a relation for each 1-cell of K (to see this consider the standard presentation for the dual triangulation of K). Letting F be the fundamental domain of $S^1 \times S^2$ given by gluing the bottom faces of the two simplices together, one easily checks that the generator of $\pi_1(M) = \mathbb{Z}$ is given by the self gluing of the first simplex taking the face opposite the third vertex to the face opposite the zeroth. For $\alpha \in \mathrm{SL}(2, \mathbb{C})$, the representation given by taking the generator to α has a decoration as

in Figure 10. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $c(A) = c$, and note that $\det(e_1, Ae_1) = c(A)$. Letting x , y and z denote the Ptolemy coordinates, we have

$$(10.20) \quad x = c(\alpha), \quad y = c(\alpha^2) = x \operatorname{Tr}(\alpha), \quad z = c(\alpha^3) = x(\operatorname{Tr}(\alpha)^2 - 1),$$

and it follows that the Ptolemy variety detects all representations except those where $\operatorname{Tr}(\alpha) = \pm 1$.

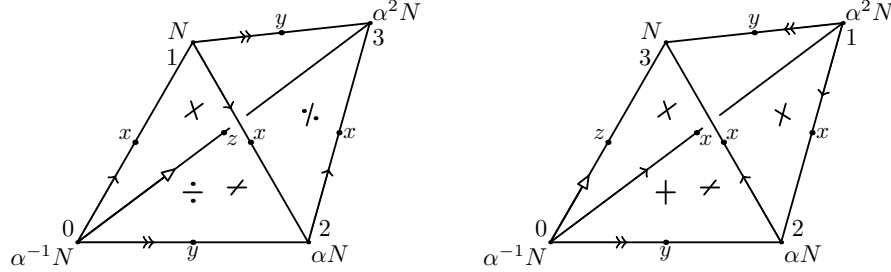


FIGURE 10. A triangulation of $S^1 \times S^2$. Both simplices have self gluings.

Remark 10.4. When $n = 2$, examples of Conjecture 1.15 are abundant. E.g. for the 10_{155} knot complement (10 simplices), the volumes of the representations detected by the Ptolemy variety are (numerically)

$$(10.21) \quad \operatorname{Vol}(m032(6, 1)), \quad 2 \operatorname{Vol}(4_1), \quad 3 \operatorname{Vol}(10_{155}) - 4 \operatorname{Vol}(v3461), \quad \operatorname{Vol}(10_{155}).$$

Remark 10.5. For the census manifolds, the Ptolemy varieties (after fixing the action by diagonal matrices) tend to be zero dimensional. Higher dimensional components do occur, but the complex volume is constant on components.

Remark 10.6. If the face pairings do not respect the vertex orderings, one can still define a Ptolemy variety by introducing more signs. This is a purely cosmetic issue, which we shall not discuss here. One can always replace a triangulation by an ordered one.

11. THE IRREDUCIBLE REPRESENTATIONS OF $\operatorname{SL}(2, \mathbb{C})$

Let $\phi_n: \operatorname{SL}(2, \mathbb{C}) \rightarrow \operatorname{SL}(n, \mathbb{C})$ denote the canonical irreducible representation. It is induced by the Lie algebra homomorphism $\mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{sl}(n, \mathbb{C})$ given by

$$(11.1) \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mapsto \operatorname{diag}^+(n-1, \dots, 1), \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mapsto \operatorname{diag}^-(1, \dots, n-1), \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mapsto \operatorname{diag}(n-1, n-3, \dots, -n+1),$$

where $\operatorname{diag}^+(v)$ and $\operatorname{diag}^-(v)$ denote matrices whose first upper (resp. lower) diagonal is v and all other entries are zero. One has

$$(11.2) \quad \phi_n \left(\begin{bmatrix} 0 & -a^{-1} \\ a & 0 \end{bmatrix} \right) = q(a^{n-1}, -a^{n-3}, \dots, (-1)^{n-1} a^{-(n-1)})$$

$$(11.3) \quad \phi_n \left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) = \pi_{n-1}(x, \dots, x) \pi_{n-2}(x, \dots, x) \cdots \pi_1(x).$$

Lemma 11.1. Let c be a Ptolemy cochain on Δ_2^3 , and let τ denote the corresponding cocycle. The assignment

$$(11.4) \quad \phi_n(c): \dot{T}^3(n) \rightarrow \mathbb{C}^*, \quad t \mapsto \phi_n(c)_t = \prod_{i < j} c_{ij}^{t_i t_j}$$

is a Ptolemy cochain on Δ_n^3 . If c is a $\operatorname{PSL}(2, \mathbb{C})$ -Ptolemy cochain with obstruction cocycle σ , $\phi_n(c)$ is a $p\operatorname{SL}(n, \mathbb{C})$ -Ptolemy cochain with obstruction cocycle σ . Moreover, $\phi_n(c)$ is the Ptolemy cochain corresponding to $\phi_n(\tau)$.

Proof. Let $\alpha = (a_0, \dots, a_3) \in T^3(n-2)$. Letting $k_\alpha = \prod_{i < j} c_{ij}^{a_i a_j}$, and $l_\alpha = \prod_{i < j} c_{ij}^{a_i + a_j}$, we have

$$(11.5) \quad \phi_n(c)_{\alpha_{03}} \phi_n(c)_{\alpha_{12}} = k_\alpha^2 l_\alpha c_{03} c_{12}, \quad \phi_n(c)_{\alpha_{01}} \phi_n(c)_{\alpha_{23}} = k_\alpha^2 l_\alpha c_{01} c_{23}, \quad \phi_n(c)_{\alpha_{02}} \phi_n(c)_{\alpha_{13}} = k_\alpha^2 l_\alpha c_{02} c_{13}.$$

Hence, the appropriate Ptolemy relations are satisfied, proving the first two statements. The long and short edges of the cocycle corresponding to $\phi_n(c)$ are given by (9.9) and (9.20), and we must prove that these agree with those of $\phi_n(\tau)$. For the long edges, this follows immediately from (11.2). For the short edges, an easy computation shows that all the diamond coordinates of a face are equal, and equal to the corresponding diamond coordinate of c . For example, the type 1 diamond coordinate on face 3 whose left vertex is $t = (t_0, t_1, t_2, 0)$ is given by

$$(11.6) \quad \frac{\phi_n(c)_{t+(0,-1,1,0)} \phi_n(c)_{t+(-1,1,0,0)}}{\phi_n(c)_t \phi_n(c)_{t+(-1,0,1,0)}} = \frac{c_{01}^{t_0(t_1-1)} c_{02}^{t_0(t_2+1)} c_{12}^{(t_1-1)(t_2+1)} c_{01}^{(t_0-1)(t_1+1)} c_{02}^{(t_0-1)t_2} c_{12}^{(t_1+1)t_2}}{c_{01}^{t_0 t_1} c_{02}^{t_0 t_2} c_{12}^{t_1 t_2} c_{01}^{(t_0-1)t_1} c_{02}^{(t_0-1)(t_2+1)} c_{12}^{t_1(t_2+1)}} \\ = \frac{c_{02}}{c_{01} c_{12}},$$

which is a diamond coordinate for c . By (11.3) the short edges thus agree with those of $\phi_n(\tau)$, proving the result. \square

Corollary 11.2. If a representation $\rho: \pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is detected by $P_2^\sigma(K)$ then $\phi_{2k+1} \circ \rho$ is detected by $P_{2k+1}(K)$ and $\phi_{2k} \circ \rho$ is detected by $P_{2k}^\sigma(K)$. \square

Theorem 11.3. Let ρ be a boundary-unipotent representation in $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{PSL}(2, \mathbb{C})$. The extended Bloch group element of $\phi_n \circ \rho$ is $\binom{n+1}{3}$ times that of ρ .

Proof. By refining the triangulation if necessary, we may represent ρ by a Ptolemy cochain c on K . Then $\phi = \phi_n(c)$ is a Ptolemy cochain representing $\phi_n \circ \rho$, and the extended Bloch group element of $\phi_n \circ \rho$ is given by

$$(11.7) \quad [\phi_n(\rho)] = \sum_i \epsilon_i \sum_{\alpha \in T^3(n-2)} (\tilde{\phi}_{\alpha_{03}}^i + \tilde{\phi}_{\alpha_{12}}^i - \tilde{\phi}_{\alpha_{02}}^i - \tilde{\phi}_{\alpha_{13}}^i, \tilde{\phi}_{\alpha_{01}}^i + \tilde{\phi}_{\alpha_{23}}^i - \tilde{\phi}_{\alpha_{02}}^i - \tilde{\phi}_{\alpha_{13}}^i).$$

By Proposition 7.7, we may choose the logarithms independently as long as we use the same logarithm for identified points. Defining $\tilde{\phi}_t^i = \sum_{j < k} t_j t_k \tilde{c}_{jk}^i$, we see that

$$(11.8) \quad (\tilde{\phi}_{\alpha_{03}}^i + \tilde{\phi}_{\alpha_{12}}^i - \tilde{\phi}_{\alpha_{02}}^i - \tilde{\phi}_{\alpha_{13}}^i, \tilde{\phi}_{\alpha_{01}}^i + \tilde{\phi}_{\alpha_{23}}^i - \tilde{\phi}_{\alpha_{02}}^i - \tilde{\phi}_{\alpha_{13}}^i) = (\tilde{c}_{03} + \tilde{c}_{12} - \tilde{c}_{02} - \tilde{c}_{13}, \tilde{c}_{01} + \tilde{c}_{23} - \tilde{c}_{02} - \tilde{c}_{13}),$$

which means that the flattenings assigned to each subsimplex of Δ_n^i are equal. By Lemma 5.6, $|T^3(n-2)| = \binom{n+1}{3}$, and the result follows. \square

11.1. Essential edges.

Definition 11.4. An edge of K is *essential* if the lifts to L have distinct end points.

Note that an edge may be essential even though it is homotopically trivial in K . Let $L^{(0)}$ denote the zero skeleton of L .

Lemma 11.5. Let ρ be a representation in $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{PSL}(2, \mathbb{C})$. A decoration of ρ determines a ρ -equivariant map

$$(11.9) \quad D: L^{(0)} \rightarrow \partial \overline{\mathbb{H}}^3 = \mathbb{C} \cup \{\infty\}, \quad e_i \mapsto g_i \infty.$$

Every such map comes from a decoration, and the decoration is generic if and only if the vertices of each simplex of L map to distinct points in $\mathbb{C} \cup \{\infty\}$.

Proof. Equivariance of (11.9) follows from the definition of a decoration. A ρ -equivariant map $D: L^{(0)} \rightarrow \mathbb{C} \cup \{\infty\}$ is uniquely determined by its image of lifts $\tilde{e}_i \in L$ of the zero cells e_i of K . Picking g_i such that $g_i\infty = D(\tilde{e}_i)$, we define a decoration by assigning the coset g_iN to \tilde{e}_i . The last statement follows from the fact that $\det(g_1e_1, g_2e_1) = 0$ if and only if $g_1\infty = g_2\infty$. \square

In the following we assume that the interior of M is a cusped hyperbolic 3-manifold \mathbb{H}^3/Γ with finite volume.

Proposition 11.6. *If all edges of K are essential, the geometric representation has a generic decoration.*

Proof. We identify $\pi_1(M)$ with $\Gamma \subset \text{PSL}(2, \mathbb{C})$. Each vertex of L corresponds to either a cusp of M or an interior point of M . Accordingly, we have $L^{(0)} = L_{\text{cusp}}^{(0)} \cup L_{\text{int}}^{(0)}$. Each point in $L_{\text{cusp}}^{(0)}$ determines a parabolic subgroup of $\text{PSL}(2, \mathbb{C})$ stabilizing a unique point in $\mathbb{C} \cup \{\infty\}$. We thus have an equivariant map $D: L_{\text{cusp}}^{(0)} \rightarrow \mathbb{C} \cup \{\infty\}$ taking a point to its stabilizer. Let e_1 and e_2 be points in $L_{\text{cusp}}^{(0)}$ connected by an edge. Since all edges of K are essential, $e_1 \neq e_2$. It is well known that the point stabilizers of different cusps are distinct. Hence, $D(e_1) \neq D(e_2)$ if e_1 and e_2 correspond to different cusps. If e_1 and e_2 correspond to the same cusp, there exists an element in Γ taking e_1 to e_2 . Since only peripheral elements (i.e. cusp stabilizers) have fixed points in $\mathbb{C} \cup \{\infty\}$, it follows that $D(e_1) \neq D(e_2)$. We extend D to $L^{(0)}$ by choosing any equivariant map $L_{\text{int}}^{(0)} \rightarrow \mathbb{C} \cup \{\infty\}$. Since such map is uniquely determined by finitely many values (which may be chosen freely), we can pick the extension so that the vertices of each simplex map to distinct points. This proves the result. \square

Theorem 11.7. *Suppose all edges of K are essential. The representation $\phi_n \circ \rho_{\text{geo}}$ is detected by $P_n(K)$ if n is odd, and by $P_n^{\sigma_{\text{geo}}}(K)$ if n is even.*

Proof. By Proposition 11.6, $P_2^{\sigma_{\text{geo}}}(K)$ detects ρ_{geo} . The result now follows from Corollary 11.2. \square

12. GLUING EQUATIONS AND PTOLEMY COCHAINS

In this section we discuss the relation between Ptolemy cochains and solutions to the gluing equations. The latter were invented by Thurston [25] to explicitly compute the hyperbolic structure (and its deformations) of a triangulated hyperbolic manifold, and used effectively in [20, 15, 9]. The gluing equations make sense for any 3-cycle. They are defined by assigning a *cross-ratio* $z_i \in \mathbb{C} \setminus \{0, 1\}$ to each simplex Δ_i of K . Given these, we assign cross-ratio parameters to the edges of Δ_i as in Figure 11.

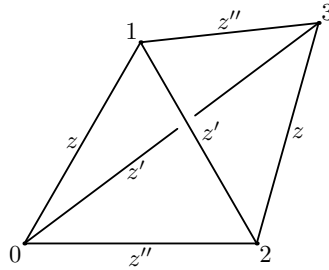


FIGURE 11. Assigning cross-ratio parameters to the edges of Δ_i . By definition, $z' = \frac{1}{1-z}$ and $z'' = 1 - \frac{1}{z}$.

There is a gluing equation for each edge E in K and each generator γ of the fundamental group of each boundary component of M . These are given by

$$(12.1) \quad \prod_{e \mapsto E} z(e)^{\epsilon_i(e)} = 1, \quad \prod_{\gamma \text{ passes } e} z(e)^{\epsilon_i(e)} = 1.$$

Here $z(e)$ denotes the cross-ratio parameter assigned to e , and $\epsilon_i(e) = \epsilon_i$ if e is an edge of Δ_i . It follows that the set of assignments $\Delta_i \mapsto z_i \in \mathbb{C} \setminus \{0, 1\}$ satisfying the gluing equations (12.1) is an algebraic set $V(K)$.

Lemma 12.1. For every point $\{z_i\} \in V(K)$ there is a map $D: L^{(0)} \rightarrow \mathbb{C} \cup \{\infty\}$ such that if $\tilde{\Delta}_i$ is a lift of Δ_i with vertices e_1, \dots, e_3 in L , the cross-ratio of the ideal simplex with vertices $D(e_1), \dots, D(e_3)$ is z_i . It is unique up to multiplication by an element in $\mathrm{PSL}(2, \mathbb{C})$. Moreover, there is a unique (up to conjugation) boundary-unipotent representation $\pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ such that D is ρ -equivariant.

Proof. Pick a fundamental domain F for K in L . Pick a simplex Δ in F and define D by mapping the first 3 vertices of Δ to 0, ∞ and 1. The map D is now uniquely determined by the cross-ratios. The fundamental group of M has a presentation with a generator for each face pairing of F . The second statement thus follows from the fact that $\mathrm{PSL}(2, \mathbb{C})$ is 3-transitive. We leave the details to the reader. \square

Given a Ptolemy cochain on K , we assign the cross-ratio $z_i = \frac{c_{03}^i c_{12}^i}{c_{02}^i c_{13}^i}$ to Δ_i . Note that the Ptolemy relations imply that the cross-ratio parameters are given by

$$(12.2) \quad z_i = \frac{c_{03}^i c_{12}^i}{c_{02}^i c_{13}^i}, \quad z'_i = \frac{c_{02}^i c_{13}^i}{c_{01}^i c_{23}^i}, \quad z''_i = -\frac{c_{01}^i c_{23}^i}{c_{03}^i c_{12}^i}.$$

Theorem 12.2. *There is a surjective regular map*

$$(12.3) \quad \coprod_{\sigma \in H^2(K; \mathbb{Z}/2\mathbb{Z})} P_2^\sigma(K) \rightarrow V(K), \quad c \mapsto \{z_i = \frac{c_{03}^i c_{12}^i}{c_{02}^i c_{13}^i}\}.$$

The fibers are all $(\mathbb{C}^)^h$, where h is the number of zero-cells of K .*

Proof. By Zickert [28, Theorem 6.5], the log-parameters (7.2) of a Ptolemy cochain satisfy the logarithmic version of the gluing equations (12.1). Note that the log-parameter of an edge is a logarithm of the corresponding cross-ratio parameter except when the edge is 02 or 13, where it is a logarithm of minus the cross-ratio parameter. As explained in the proof of Proposition 7.7, any curve passes these an even number of times. It thus follows that the cross-ratios satisfy the gluing equations. Surjectivity follows from Lemma 11.5, and the fact that fibers are $(\mathbb{C}^*)^h$ follows from the fact that $g_1 \infty = g_2 \infty$ if and only if $g_1 N = g_2 dN$ for a unique diagonal matrix d . \square

13. OTHER FIELDS

The Ptolemy varieties $P_n(K)$ and $P_n^\sigma(K)$ may be defined over an arbitrary field F , and as in Section 9, a Ptolemy cochain determines a boundary-unipotent representation in $\mathrm{SL}(n, F)$, respectively, $p\mathrm{SL}(n, F)$. If E is an extension of F^* by \mathbb{Z} , there are maps

$$(13.1) \quad V_n(K)_F \rightarrow \widehat{\mathcal{B}}_E(F), \quad V_n^\sigma(K)_F \rightarrow \widehat{\mathcal{B}}_E(F)_{\mathrm{PSL}}$$

defined as in (5.10) using a set theoretic section of $E \rightarrow F^*$ instead of a logarithm. If F is infinite, the chain complex of Ptolemy cochains computes relative homology (see Proposition 9.5) and we

have maps

$$(13.2) \quad H_3(\mathrm{SL}(n, F)) \rightarrow \widehat{\mathcal{B}}_E(F), \quad H_3(p\mathrm{SL}(n, F)) \rightarrow \widehat{\mathcal{B}}_E(F)_{\mathrm{PSL}}.$$

It thus follows that every boundary-unipotent representation has an extended Bloch group element $[\rho]$. If F is a number field, the extended Bloch groups are independent of the extension E .

Proposition 13.1. Let F be a number field, and let $\rho: \pi_1(M) \rightarrow \mathrm{SL}(n, F)$ be a boundary-unipotent representation. If ρ is irreducible, $[\rho]$ lies in $\widehat{\mathcal{B}}(\mathrm{Tr}(\rho))$.

Proof. Let σ be an automorphism of F over $\mathrm{Tr}(\rho)$ and let $\tau: F \rightarrow \mathbb{C}$ be an embedding. Then ρ and $\sigma \circ \rho$ have the same traces, so $\tau \circ \rho$ and $\tau \circ \sigma \circ \rho$ are conjugate in $\mathrm{SL}(n, \mathbb{C})$, and thus have the same extended Bloch group element in $\widehat{\mathcal{B}}(\mathbb{C})$. By Corollary 3.6, it follows that $[\rho] = [\sigma \circ \rho] \in \widehat{\mathcal{B}}(F)$. Hence, $[\rho]$ is invariant under all automorphisms of F over $\mathrm{Tr}(\rho)$, so $[\rho] \in \widehat{\mathcal{B}}(\mathrm{Tr}(\rho))$ by Galois descent. \square

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